Erdős-Stone theorem for graphs with chromatic number 2 and 3.

In this note we will prove a special case of the Erdős-Stone theorem, which in full generality completely determines the Turán density for all graphs. A graph $H$ is $k$-colorable if there exists a function $c : V(H) \rightarrow [k]$ so that $c(v) \neq c(w)$ for every pair of adjacent vertices $v, w \in V(H)$. The chromatic number $\chi(H)$ of $H$ is the minimum positive integer $k$ such that $H$ is $k$-colorable.

**Theorem 1.** [Erdős-Stone, 1946] For every graph $H$ with at least one edge we have

$$\pi(H) = \frac{\chi(H) - 2}{\chi(H) - 1}. \quad \text{Note that the construction which is used to show that } \pi(K_t) \geq t - 2 \text{ implies that } \pi(H) \geq \frac{\chi(H) - 2}{\chi(H) - 1} \text{ for every graph } H. \text{ Thus it suffices to establish the lower bound on the Turán density. We will prove this lower bound for graphs } H \text{ with } \chi(H) = 2 \text{ and } \chi(H) = 3.

Let $K_{r,r}$ denote the complete balanced bipartite graph on $2r$ vertices, that is $V(K_{r,r}) = V_1 \cup V_2$, $|V_1| = |V_2| = r$, and $v_1v_2 \in E(K_{r,r})$ for every $v_1 \in V_1$ and $v_2 \in V_2$. The complete balanced 3-partite graph $K_{r,r,r}$ is defined similarly with $|V(K_{r,r,r})| = 3r$, $V(K_{r,r,r}) = V_1 \cup V_2 \cup V_3$, $|V_1| = |V_2| = |V_3| = r$, and $v_iv_j \in E(K_{r,r,r})$ for all $v_i \in V_i$ and $v_j \in V_j$ for $i, j \in \{1, 2, 3\}, i \neq j$. It is easy to see that a graph $H$ satisfies $\chi(H) \leq 2$ if and only if $H$ is a subgraph of $K_{r,r}$ for some $r$, and similarly, $\chi(H) \leq 3$ if and only if $H$ is a subgraph of $K_{r,r,r}$ for some $r$. Therefore to establish Theorem 1 in the cases we are interested in it suffices to show that $\pi(K_{r,r}) = 0$ and $\pi(K_{r,r,r}) = 1/2$ for every positive integer $r$. The next lemma establishes the first of these identities.

For a graph $G$ and a vertex $v \in V(G)$ let $N(v)$ denote the neighborhood of $v$, that is the set of vertices of $G$ adjacent to $v$. 
Lemma 2. For every positive integer $r$ and every $\varepsilon > 0$ there exists $n_0 > 0$ so that every graph $G$ with $n := |V(G)| \geq n_0$ and $E(G) \geq \varepsilon n^2$ has $K_{r,r}$ as a subgraph.

Proof. Note that it suffices to show that there exist distinct vertices $v_1, v_2, \ldots, v_r \in V(G)$ such that

$$|N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)| \geq r.$$ 

We will show that for an appropriate choice of $n_0$ one has

$$\sum_{\{v_1, \ldots, v_r\} \subseteq V(G)} |N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)| \geq rn^r.$$ 

The lemma will follow by averaging. For some constant $c_r$ depending only on $r$ we have

$$\sum_{\{v_1, \ldots, v_r\} \subseteq V(G)} |N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)| = \sum_{w \in V(G)} \left( \frac{|N(w)|}{r} \right)$$

$$\geq \sum_{w \in V(G)} \left( \frac{1}{r!} \deg^r(w) - c_r \deg^{r-1}(w) \right)$$

$$\geq \frac{1}{r!} \left( \sum_{w \in V(G)} \deg^r(w) \right) - c_r n^r$$

$$\geq \frac{n}{r!} \left( \frac{\sum_{w \in V(G)} \deg(w) \deg^r(w)}{n} \right)^r - c_r n^r$$

$$\geq \frac{n(\varepsilon n/2)^r}{r!} - c_r n^r \geq rn^r,$$

as desired. \qed

The next technical lemma will allow us to prove $\pi(K_{r,r,r}) = 0$ amplifying the result of Lemma 2. The proof is left as an exercise.

Lemma 3. Let $H$ be a fixed $s$-graph of order $k$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 > 0$ with the following properties. If $G$ is an $s$-graph of order $n \geq n_0$ with $|G| \geq (\pi(H) + \varepsilon) \binom{n}{s}$ then at least $\delta \binom{n}{k}$ subsets of $V(G)$ of size $k$ induce an $s$-graph containing $H$. 
In this note we will only use Lemma 3 when $H$ is a 2-graph. Let $K^+_2$ denote a graph on $r + 2$ vertices with two of the vertices adjacent two each other and all the remaining vertices, and no other edges.

**Lemma 4.** $\pi(K^+_2, r) = 1/2$.

**Proof.** Suppose for a contradiction that $\pi(K^+_2, r) \geq 1/2 + 2\varepsilon$ for some $\varepsilon > 0$. Let $\delta$ and $n_0$ be chosen to satisfy Lemma 3 for $H = K_3$ and $\varepsilon$. Let $n' := \max\{n_0, 4r/\delta\}$ and let $G$ be a graph of order $n \geq n'$ with $|G| \geq (1/2 + \varepsilon)(n\choose 2)$.

By Lemma 3, $G$ contains at least $\delta(n\choose 3) \geq 3r(n\choose 2)$ triangles, where the inequality holds by the choice of $n'$. It follows that some edge of $G$ belongs to at least $r$ triangles. Thus $G$ contains a copy of $K^+_2$. It follows that $\pi(K^+_2, r) \leq 1/2 + \varepsilon$, contradicting the choice of $\varepsilon$. □

**Lemma 5.** $\pi(K_{r,r,r}) = 1/2$.

**Proof.** The proof follows the pattern of the proof of Lemma 4. Suppose for a contradiction that $\pi(K_{r,r,r}) \geq 1/2 + 2\varepsilon$ for some $\varepsilon > 0$. Let $\delta$ and $n_0$ be chosen to satisfy Lemma 3 for $H = K^+_2$ and $\varepsilon$. Let $n'$ be sufficiently large, which will be chosen implicitly later, depending on $\delta, n_0$ and $r$, so that $n' \geq n_0$, in particular. Let $G$ be a graph of order $n \geq n'$ with $|G| \geq (1/2 + \varepsilon)(n\choose 2)$.

For $\{v_1, v_2, \ldots, v_r\} \subseteq V(G)$, let $e_N(v_1, v_2, \ldots, v_r)$ denote the number of edges of $G$ joining pairs of vertices in $N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)$. By Lemma 3, we have

$$\sum_{\{v_1, v_2, \ldots, v_r\} \subseteq V(G)} e_N(v_1, v_2, \ldots, v_r) \geq \frac{\delta}{r^2} \binom{n}{r + 2} \geq \frac{\delta n^2}{r^4} \binom{n}{r}.$$

It follows that $e_N(v_1, v_2, \ldots, v_r) \geq \delta n^2/r^4$ for some $\{v_1, v_2, \ldots, v_r\} \subseteq V(G)$. We now apply Lemma 2 with $\delta/r^4$ instead of $\varepsilon$ and we assume that $n'$ has been chosen large enough to satisfy the conclusion of this lemma. It now follows that $N(v_1) \cap N(v_2) \cap \ldots \cap N(v_r)$ contains a copy of $K_{r,r}$. Thus $G$
contains a copy of $K_{r,r,r}$. It follows that $\pi(K_{r,r,r}) \leq 1/2 + \varepsilon$, contradicting the choice of $\varepsilon$. \qed