Assignment #1: Sets and Logic. Solutions.

1. **Set Identities.** Prove the following set identities.

   a) \((A - B) \cap C = (A \cap C) - (B \cap C)\),

   b) \((A \oplus B) \oplus (A \oplus C) = B \oplus C\).

**Solution a):** Consider \(x \in (A - B) \cap C\). We have \(x \in A - B\) and \(x \in C\). Therefore, \(x \in A\) and \(x \notin B\). It follows that \(x \in A \cap C\) and \(x \notin B \cap C\). Therefore, \(x \in (A \cap C) - (B \cap C)\).

For the other inclusion consider \(y \in (A \cap C) - (B \cap C)\). We have \(y \in A \cap C\) and \(y \notin B \cap C\). Therefore, \(y \in A\) and \(y \in C\). As \(y \notin B \cap C\), it follows that \(y \notin B\). Therefore \(y \in (A - B) \cap C\).

We have shown that every element of \((A - B) \cap C\) is an element of \((A \cap C) - (B \cap C)\) and vice versa. Therefore the sets are equal.

**b):** The argument is similar to that in a). Consider \(x \in (A \oplus B) \oplus (A \oplus C)\). It follows that \(x\) belongs to exactly one of the sets \(A \oplus B\) and \(A \oplus C\). Both sides of the formula are symmetric with respect to switching the roles of \(B\) and \(C\), so without loss of generality we suppose that \(x \in A \oplus B\) and \(x \notin A \oplus C\). Therefore, \(x\) is in exactly one of the sets \(A\) and \(B\). If \(x \in A\) and \(x \notin B\) then \(x \in C\), as \(x \notin A \oplus C\). In this case, \(x \in B \oplus C\). In the remaining case \(x \notin A\) and \(x \in B\), we have \(x \notin C\). Again, we conclude that \(x \in B \oplus C\).

The proof of the other inclusion is analogous.

2. **Propositions.** Which of the following sentences are propositions?

   a) It rained yesterday.

   b) The last digit of the smallest prime number larger than \(100^{100}\) is 1.

   c) This sentence is false.

   d) \(6 + 5 = 10\).

**Solution:** All but c) are propositions as each of them is either true or false. The sentence in c) can be neither true nor false.
3. Truth tables. Use a truth table to verify the following equivalence

\[-(P \lor (Q \land \neg R)) \leftrightarrow (\neg P) \land ((\neg Q) \lor R).\]

Solution:

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<th>Q</th>
<th>R</th>
<th>Q \land (\neg R)</th>
<th>\neg (P \lor (Q \land (\neg R)))</th>
<th>(\neg Q) \lor R</th>
<th>(\neg P) \land ((\neg Q) \lor R)</th>
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4. Conditional equivalence. Which of the following implications are true?
   
   a) If 2 + 2 = 5 then 2 + 2 = 6.
   
   b) If 2 + 2 = 4 then the world is flat.
   
   c) If both of the previous statements are true then 2 + 2 = 7.

Solution: a) True. b) False. c) True.

5. Tautologies. Which of the following are tautologies? If the statement is a tautology give a proof using the appropriate rules of logic demonstrated in class at each step of the proof. (Avoid using truth tables if possible.) If not, then justify your answer by giving a counter-example, i.e., a truth assignment which results in a false value.

   a) \( p \rightarrow (p \lor q). \)
   
   b) \((p \lor q \lor r) \land (p \rightarrow r) \land (q \rightarrow r)) \rightarrow r. \)
   
   c) \((p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \rightarrow q) \rightarrow r). \)
Solution a):

\[
\begin{align*}
  p & \rightarrow (p \lor q) \iff \\
  (\neg p) & \lor (p \lor q) \iff \\
  (\neg p \lor p) & \lor q \iff \\
  1 & \lor q \iff 1.
\end{align*}
\]

Solution b): If \( r \) is true then the implication is true so assume \( r \) is false. The formula is now true as long as

\[
(p \lor q \lor 0) \land (p \rightarrow 0) \land (q \rightarrow 0) \iff 0.
\]

But

\[
\begin{align*}
  (p \lor q \lor 0) & \land (p \rightarrow 0) \land (q \rightarrow 0) \iff \\
  (p \lor q) & \land \neg p \land \neg q \iff \\
  (p \lor q) & \land \neg (p \lor q) \iff \\
  0,
\end{align*}
\]

as desired.

Solution c): Let \( p \) and \( r \) be false and let \( q \) be true. Then \( q \rightarrow r \) is false, and \( p \rightarrow q \) is true. The left side of the equivalence is then true and the right side is false. Thus the logic formula is not a tautology.

6. **Circuits.** Suppose we have a committee of four people. Design a circuit which determines if exactly two of them vote yes on an issue.

Solution: There are many solutions. The most straightforward one is the circuit corresponding to the following logic formula:

\[
\begin{align*}
  (p \land q \land \neg r \land \neg s) & \lor (p \land \neg q \land r \land \neg s) \lor (p \land \neg q \land \neg r \land s) \lor \\
  (p \land q \land \neg r \land \neg s) & \lor (\neg p \land q \land \neg r \land \neg s) \lor (\neg p \land \neg q \land \neg r \land s) \lor \\
  (\neg p \land q \land \neg r \land \neg s) & \lor (\neg p \land \neg q \land \neg r \land \neg s).
\end{align*}
\]