Counting using bijections

Review of MATH 240. A function \( f : X \to Y \) is

- a surjection (or onto) if for every \( y \in Y \) there exists \( x \in X \) such that \( y = f(x) \),
- an injection if for every \( y \in Y \) there exists at most one \( x \in X \) such that \( y = f(x) \),
- a bijection if for every \( y \in Y \) there exists exactly one \( x \in X \) such that \( y = f(x) \).

Let \([n]\) denote \( \{1, 2, \ldots, n\} \)

**Theorem 1.**

1. There exist \( n^k \) sequences \( s_1s_2\ldots s_k \) of length \( k \) such that \( s_1, s_2, \ldots, s_k \in [n] \). Equivalently, there are \( n^k \) functions \( f : [k] \to [n] \).
2. There exist \( n(n-1)\ldots(n-k+1) \) sequences \( s_1s_2\ldots s_k \) of length \( n \) such that \( s_1, s_2, \ldots, s_k \in [n] \), and \( s_i \neq s_j \) for \( i \neq j \). Equivalently, there are \( n(n-1)\ldots(n-k+1) \) injections \( f : [k] \to [n] \).
3. There are \( n! \) permutations of \([n]\), i.e. bijections \( f : [n] \to [n] \).

**Lemma 2.** There are \( 2^n \) subsets of an \( n \) element set.

**Lemma 3.** There are

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

\( k \) element subsets of an \( n \) element set.

**Theorem 4** (Binomial theorem).

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]

**Corollary 5.**

\[
2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n},
\]

\[
\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots
\]

**Theorem 6.** There exist \( \binom{n+k-1}{k-1} \) solutions to the equation

\[
x_1 + x_2 + \ldots + x_k = n,
\]

such that \( x_1, x_2, \ldots, x_k \geq 0 \) are integers.

There exist \( \binom{n-1}{k-1} \) solutions to the above equation if we require that \( x_1, x_2, \ldots, x_k \geq 1 \) instead.

**Labelled trees.**

**Theorem 7.** There exist \( n^{n-2} \) trees on \( n \) vertices with vertices labelled 1, 2, \ldots, \( n \).
Catalan numbers. Let \( C_n \) denote the \( n \)th Catalan number.

\[
C_n = \frac{1}{n+1} \binom{2n}{n}.
\]

**Theorem 8.** \( C_n \) counts the number of the following objects:

- Sequences of \( n \) pluses and \( n \) minuses, such that each (initial) partial sum is non-negative.
- Dyck walks: Paths from \((0,0)\) to \((2n,0)\) using steps \((1,1)\) and \((1,-1)\) and never going below the \( x \) axis.
- Rooted plane trees with \( n+1 \) vertices.
- Planted\(^1\) trivalent\(^2\) with \( 2n+2 \) vertices
- Decompositions of an \((n+2)\)-gon into \( n \) triangles.

**Generating functions**

The formal power series

\[
F(x) = \sum_{n \geq 0} f(n)x^n
\]

is the ordinary generating function for the sequence \( f(n) \).

**Basic generating function method.**

1. Find a recurrence for \( f(n) \)
2. Multiply both sides of the recurrence by \( x^n \).
3. Solve the resulting equation to find \( F(x) \).
4. Express \( F(x) \) as power series again to find \( f(n) \).

\[
\frac{1}{1-ax} = \sum_{n \geq 0} a^n x^n.
\]

**Manipulating ordinary generating functions.** Let \( F(x) = \sum_{n \geq 0} f(n)x^n \) then

\[
\sum_{n \geq 0} f(n+k)x^n = \frac{F(x) - f(0) - f(1)x - \ldots - f(k-1)x^{k-1}}{x^k},
\]

\[
x \frac{d}{dx} F(x) = \sum_{n \geq 0} n f(n)x^n.
\]

If \( G(x) = \sum_{n \geq 0} g(n)x^n \), \( H(x) = \sum_{n \geq 0} h(n)x^n \) then

\[
H(x)G(x) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} h(k)g(n-k) \right) x^n.
\]

**Convolutions.** If \( F_i(x) \) is the generating function for selecting items from the set \( S_i \) for \( i = 1, 2, \ldots, k \) then \( F_1(x)F_2(x) \ldots F_k(x) \) is the generating functions for selecting items from \( S_1 \cup S_2 \ldots \cup S_k \).

\(^1\)the root has degree one
\(^2\)every vertex has degree one or three
Exponential generating functions. The formal power series
\[ \hat{F}(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!} \]
is the exponential generating function for the sequence \( f(n) \).

If \( \hat{F}(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!} \) then
\[ \frac{d}{dx} \hat{F}(x) = \sum_{n \geq 0} f(n+1) \frac{x^n}{n!} , \]
\[ x \frac{d}{dx} \hat{F}(x) = \sum_{n \geq 0} n f(n) \frac{x^n}{n!} , \]
If \( G(x) = \sum_{n \geq 0} g(n) \frac{x^n}{n!} \), \( H(x) = \sum_{n \geq 0} h(n) \frac{x^n}{n!} \) then
\[ \hat{H}(x)\hat{G}(x) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} h(k) g(n-k) \right) \frac{x^n}{n!} . \]
\[ e^{ax} = \sum_{n \geq 0} a^n \frac{x^n}{n!} . \]