Classical results.

1. **Fermat’s little theorem.** Let \( p \) be a prime number, and \( n \) a positive integer. Show that \( n^p - n \) is divisible by \( p \).

2. An **Hadamard matrix** is an \( n \times n \) square matrix, all of whose entries are \(+1\) or \(-1\), such that every pair of distinct rows is orthogonal. In other words, if the rows are considered to be vectors of length \( n \), then the dot product between any two distinct row-vectors is zero. Show that there exist infinitely many Hadamard matrices.

3. **Ramsey’s theorem.** Show that for any pair of positive integers \((r, s)\), there exists a positive integer \( R(r, s) \) such that if the edges of a complete graph on \( R(r, s) \) vertices are coloured red or blue, then either there exists a complete subgraph on \( r \) vertices which is entirely blue, or a complete subgraph on \( s \) vertices which is entirely red. (A complete graph is a graph where every two vertices are connected by an edge.)

Problems.

1. Let \( n \) be a positive integer. Prove that

   \[
   1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{3}{2}.
   \]

2. **Putnam 2008. B2.** Let \( F_0(x) = \ln x \). For \( n \geq 0 \) and \( x > 0 \), let \( F_{n+1}(x) = \int_0^x F_n(t) \, dt \). Evaluate

   \[
   \lim_{n \to \infty} \frac{n!F_n(1)}{\ln n}.
   \]

3. **GA 32.** Show that if \( a_1, a_2, \ldots, a_n \) are non-negative real numbers, then

   \[
   (1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq (1 + \sqrt[n]{a_1 a_2 \cdots a_n})^n.
   \]

4. **IMO 2001. B1.** 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

5. **Putnam 2004. A3.** Define a sequence \( \{u_n\}_{n=0}^{\infty} \) by \( u_0 = u_1 = u_2 = 1 \), and thereafter by the condition that

   \[
   \det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!
   \]

   for all \( n \geq 0 \). Show that \( u_n \) is an integer for all \( n \). (By convention, \( 0! = 1 \).)

6. **Putnam 2015. B2.** Given a list of the positive integers 1, 2, 3, 4, \ldots, take the first three numbers 1, 2, 3 and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers 4, 5, 7 and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and consider the sequence of sums produced: 6, 16, 27, 36, \ldots. Prove or disprove that there is some number in the sequence whose base 10 representation ends with 2015.
7. **Putnam 1996. A4.** Let $S$ be the set of ordered triples $(a, b, c)$ of distinct elements of a finite set $A$. Suppose that

(a) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;

(b) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$;

(c) $(a, b, c)$ and $(c, d, a)$ are both in $S$ if and only if $(b, c, d)$ and $(d, a, b)$ are both in $S$.

Prove that there exists a one-to-one function $g$ from $A$ to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

8. **Putnam 1996. A6.** A triangulation $T$ of a polygon $P$ is a finite collection of triangles whose union is $P$, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side is a side of exactly one triangle in $T$. Say that $T$ is admissible if every internal vertex is shared by 6 or more triangles. Prove that there is an integer $M_n$, depending only on $n$, such that any admissible triangulation of a polygon $P$ with $n$ sides has at most $M_n$ triangles.