Problem Seminar.
Number theory.

Classical results.

1. **Euler.** For a positive integer $n$ and any integer $a$ relatively prime to $n$ one has
   \[ a^{\phi(n)} \equiv 1 \pmod{n}, \]
   where $\phi(n)$ is the number of positive integers between 1 and $n$ relatively prime to $n$.

2. **Polignac’s formula.** If $p$ is a prime number and $n$ a positive integer, then the exponent of $p$ in $n!$ is
   \[ \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots. \]

3. **Chinese Remainder theorem.** Let $m_1, m_2, \ldots, m_k$ be pairwise positive integers greater than 1, such that $\gcd(m_i, m_j) = 1$ for $i \neq j$. Then for any integers $a_1, a_2, \ldots, a_k$ the system of congruences
   \[ x \equiv a_1 \pmod{m_1}, \]
   \[ x \equiv a_2 \pmod{m_2}, \]
   \[ \ldots \]
   \[ x \equiv a_k \pmod{m_k}. \]
   has solutions, and any two such solutions are congruent modulo $m = m_1 m_2 \ldots m_k$.

4. **Sylvester’s theorem.** Let $a$ and $b$ be positive integers with $\gcd(a, b) = 1$. Then $ab - a - b$ is the largest positive integer $c$ for which the equation $ax + by = c$ is not solvable in nonnegative integers.

Problems.

1. Prove that $n!$ is not divisible by $2^n$ for any positive integer $n$.

2. **Putnam 1956. A2.** Given any positive integer $n$, show that we can find a positive integer $m$ such that $mn$ uses all ten digits when written in the usual base 10.

3. **Putnam 2000. A2.** Prove that there exist infinitely many integers $n$ such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

4. **Putnam 2013. A2.** Let $S$ be the set of all positive integers that are not perfect squares. For $n$ in $S$, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of $a_r$ over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4, 2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function $f$ from $S$ to the integers is one-to-one.
5. **Putnam 2000. B2.** Prove that the expression
\[
gcd(m, n)^{n \choose m}
\]
is an integer for all pairs of integers \( n \geq m \geq 1 \).

6. **USA 1991.** Let \( n \) be an arbitrary positive integer. Show that the following sequence is eventually constant modulo \( n \):
\[
2, 2^2, 2^{2^2}, 2^{2^{2^2}}, 2^{2^{2^{2^2}}}, \ldots
\]

7. **IMO 2002.** The positive divisors of an integer \( n > 1 \) are \( 1 = d_1 < d_2 < \ldots < d_k = n \). Let \( s = d_1d_2 + d_2d_3 + \ldots + d_{k-1}d_k \). Prove that \( s < n^2 \) and find all \( n \) for which \( s \) divides \( n^2 \).

8. **IMO 2011.** Let \( f \) be a function from the set of integers to the set of positive integers. Suppose that, for any two integers \( m \) and \( n \), the difference \( f(m) - f(n) \) is divisible by \( f(m - n) \). Prove that, for all integers \( m \) and \( n \) with \( f(m) \leq f(n) \), the number \( f(n) \) is divisible by \( f(m) \).