Problem Set 3. Algebra.

Classical results.

1. **Hilbert.** Let
   \[
   H = \begin{bmatrix}
   1 & 1 & 1 & \cdots & 1 \\
   1 & 1 & 2 & \cdots & 1 \\
   1 & 3 & 1 & \cdots & 5 \\
   \vdots & \vdots & \vdots & \ddots & \vdots \\
   1 & n+1 & \frac{1}{n} & \cdots & 2n-1
   \end{bmatrix}.
   \]

   Then \(\det(H) \neq 0\).

2. Let \(p\) be a prime. Show that the polynomial \(x^{p-1} + x^{p-2} + \ldots + x + 1\) can not be expressed as a product of two non-constant polynomials with integer coefficients.

3. In Oddtown there are \(n\) citizens and \(m\) clubs \(A_1, A_2, \ldots, A_m \subseteq \{1, 2, \ldots, n\}\). The laws of Oddtown prescribe that
   - The clubs must have distinct memberships. (\(A_i \neq A_j\) for \(i \neq j\)),
   - Every club has odd number of members,
   - Every two distinct clubs have an even number of members in common. (\(|A_i \cap A_j|\) is even if \(i \neq j\)).

   Show that \(m \leq n\).

Problems.

1. **Putnam 1959. A1.** Prove that one can find a polynomial \(P(y)\) with real coefficients such that \(P(x - 1/x) = x^n - 1/x^n\) if and only if \(n\) is odd.

2. **Putnam 1991. A2.** \(M\) and \(N\) are real unequal \(n \times n\) matrices satisfying \(M^3 = N^3\) and \(M^2 N = N^2 M\). Can we choose \(M\) and \(N\) so that \(M^2 + N^2\) is invertible?

3. **Putnam 2012. A2.** Let \(*\) be a commutative and associative binary operation on a set \(S\). Assume that for every \(x\) and \(y\) in \(S\), there exists \(z\) in \(S\) such that \(x \ast z = y\). (This \(z\) may depend on \(x\) and \(y\).) Show that if \(a, b, c\) are in \(S\) and \(a \ast c = b \ast c\), then \(a = b\).

4. **Putnam 2008. A2.** Alan and Barbara play a game in which they take turns filling entries of an initially empty \(2008 \times 2008\) array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

5. **Putnam 1994. A4.** Let \(A\) and \(B\) be \(2 \times 2\) matrices with integer entries such that \(A, A + B, A + 2B, A + 3B, A + 4B\) are all invertible matrices whose inverses have integer entries. Show that \(A + 5B\) is invertible and that its inverse has integer entries.

6. **Putnam 2006. B4.** Let \(Z\) denote the set of points in \(\mathbb{R}^n\) whose coordinates are 0 or 1. (Thus \(Z\) has \(2^n\) elements, which are the vertices of a unit hypercube in \(\mathbb{R}^n\).) Let \(k\) be given, \(0 \leq k \leq n\). Find the maximum, over all vector subspaces \(V \subseteq \mathbb{R}^n\) of dimension \(k\), of the number of points in \(V \cap Z\).
7. **Putnam 2014. A6.** Let \( n \) be a positive integer. What is the largest \( k \) for which there exist \( n \times n \) matrices \( M_1, \ldots, M_k \) and \( N_1, \ldots, N_k \) with real entries such that for all \( i \) and \( j \), the matrix product \( M_i N_j \) has a zero entry somewhere on its diagonal if and only if \( i \neq j \)?

8. **Putnam 1996. B6.** The origin lies inside a convex polygon whose vertices have coordinates \((a_i, b_i)\) for \( i = 1, 2, \ldots, n \). Show that we can find \( x, y > 0 \) such that

\[
a_1 x^{a_1} y^{b_1} + a_2 x^{a_2} y^{b_2} + \ldots + a_n x^{a_n} y^{b_n} = 0
\]

and

\[
b_1 x^{a_1} y^{b_1} + b_2 x^{a_2} y^{b_2} + \ldots + b_n x^{a_n} y^{b_n} = 0.
\]