Problem Solving Seminar Fall 2021. Problem Set 7: Calculus.

Classical results.

1. Every continuous mapping of a circle into a line carries some pair of diametrically opposite points to the same point.

2. Leibniz formula.
   \[
   \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.
   \]

3. Gaussian integral.
   \[
   \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.
   \]

Problems.

1. Putnam 1964. B1. Let \(a_1, a_2, \ldots\) be positive integers such that \(\sum_{i=1}^{\infty} \frac{1}{a_i}\) converges. For each \(n\), let \(b_n\) denote the number of positive integers \(i\) for which \(a_i \leq n\). Prove that \(\lim_{n \to \infty} \frac{b_n}{n} = 0\).

2. Putnam 2012. B1. Let \(S\) be a class of functions from \([0, \infty)\) to \([0, \infty)\) that satisfies:
   (i) The functions \(f_1(x) = e^x - 1\) and \(f_2(x) = \ln(x + 1)\) are in \(S\);
   (ii) If \(f(x)\) and \(g(x)\) are in \(S\), the functions \(f(x) + g(x)\) and \(f(g(x))\) are in \(S\);
   (iii) If \(f(x)\) and \(g(x)\) are in \(S\) and \(f(x) \geq g(x)\) for all \(x \geq 0\), then the function \(f(x) - g(x)\) is in \(S\).
   Prove that if \(f(x)\) and \(g(x)\) are in \(S\), then the function \(f(x)g(x)\) is also in \(S\).

3. Putnam 1991. B2. Suppose \(f\) and \(g\) are non-constant, differentiable, real-valued functions on \(\mathbb{R}\). Furthermore, suppose that for each pair of real numbers \(x\) and \(y\),
   \[
   f(x + y) = f(x)f(y) - g(x)g(y),
   \]
   \[
   g(x + y) = f(x)g(y) + g(x)f(y).
   \]
   If \(f'(0) = 0\) prove that \((f(x))^2 + (g(x))^2 = 1\) for all real \(x\).

4. Putnam 2007. B2. Suppose that \(f : [0, 1] \to \mathbb{R}\) has a continuous derivative and that \(\int_0^1 f(x) \, dx = 0\). Prove that for every \(\alpha \in (0, 1)\),
   \[
   \left| \int_0^\alpha f(x) \, dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.
   \]

5. Putnam 2013. A3. Suppose that the real numbers \(a_0, a_1, \ldots, a_n\) and \(x\), with \(0 < x < 1\), satisfy
   \[
   \frac{a_0}{1 - x} + \frac{a_1}{1 - x^2} + \cdots + \frac{a_n}{1 - x^{n+1}} = 0.
   \]
   Prove that there exists a real number \(y\) with \(0 < y < 1\) such that
   \[
   a_0 + a_1y + \cdots + a_ny^n = 0.
   \]
6. **Putnam 2008. A4.** Define \( f: \mathbb{R} \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
  x & \text{if } x \leq e \\
  xf(\ln x) & \text{if } x > e.
\end{cases}
\]

Does \( \sum_{n=1}^{\infty} \frac{1}{f(n)} \) converge?

7. **Putnam 1993. B4.** The function \( K(x, y) \) is positive and continuous for \( 0 \leq x \leq 1, 0 \leq y \leq 1 \), and the functions \( f(x) \) and \( g(x) \) are positive and continuous for \( 0 \leq x \leq 1 \). Suppose that

\[
\int_0^1 f(y)K(x, y) \, dy = g(x) \quad \text{and} \quad \int_0^1 g(y)K(x, y) \, dy = f(x)
\]

for all \( 0 \leq x \leq 1 \). Show that \( f(x) = g(x) \) for \( 0 \leq x \leq 1 \).