Classical results.

1. Finitely many lines divide the plane into regions. Show that these regions can be colored by two colors in such a way that neighboring regions have different colors.

2. **Fermat’s little theorem.** Let $p$ be a prime number, and $n$ a positive integer. Show that $n^p - n$ is divisible by $p$.

3. An **Hadamard matrix** is an $n \times n$ square matrix, all of whose entries are $+1$ or $-1$, such that every pair of distinct rows is orthogonal. In other words, if the rows are considered to be vectors of length $n$, then the dot product between any two distinct row-vectors is zero. Show that there exist infinitely many Hadamard matrices.

4. Prove that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \ldots \pm n^2$ for some positive integer $n$ and some choice of the signs.

Problems.

1. **Putnam 2001. A2.** You have coins $C_1, C_2, \ldots, C_n$. For each $k$, $C_k$ is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the $n$ coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of $n$.

2. **GA 32.** Show that if $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then
   \[
   (1 + a_1)(1 + a_2)\ldots(1 + a_n) \geq (1 + \sqrt[n]{a_1a_2\ldots a_n})^n.
   \]

3. **IMO 2001. B1.** 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

4. **Putnam 2004. A3.** Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that
   \[
   \det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!
   \]
   for all $n \geq 0$. Show that $u_n$ is an integer for all $n$. (By convention, $0! = 1$.)

5. **Putnam 2006. B3.** Let $S$ be a finite set of points in the plane. A linear partition of $S$ is an unordered pair $\{A, B\}$ of subsets of $S$ such that $A \cup B = S$, $A \cap B = \emptyset$, and $A$ and $B$ lie on opposite sides of some straight line disjoint from $S$ ($A$ or $B$ may be empty). Let $L_S$ be the number of linear partitions of $S$. For each positive integer $n$, find the maximum of $L_S$ over all sets $S$ of $n$ points.

6. **Putnam 1996. A4.** Let $S$ be the set of ordered triples $(a, b, c)$ of distinct elements of a finite set $A$. Suppose that
   
   (a) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
   
   (b) $(a, b, c) \in S$ if and only if $(c, b, a) \not\in S$;
   
   (c) $(a, b, c)$ and $(c, d, a)$ are both in $S$ if and only if $(b, c, d)$ and $(d, a, b)$ are both in $S$.

   Prove that there exists a one-to-one function $g$ from $A$ to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

7. **Putnam 2000. B5.** Let $S_0$ be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows: the integer $a$ is in $S_{n+1}$ if and only if exactly one of $a - 1$ or $a$ is in $S_n$. Show that there exist infinitely many integers $N$ for which $S_N = S_0 \cup \{N + a : a \in S_0\}$. 