1. **Bollobás 3.9.** Suppose $A \subseteq \mathcal{P}([n])$ is an ideal, i.e. if $B \subseteq A$ and $A \in A$ then $B \in A$. Use the local LYM inequality to show that the average size of an element of $A$ is at most $n/2$.

2. Let $n$ be a positive integer. Consider a set $T_n = \{0, 1, 2\}^n$ consisting of all sequences $(a_1, a_2, \ldots, a_n)$ with $a_i \in \{0, 1, 2\}$ for $i \in [n]$. We define a partial order on $T_n$ so that $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ if and only if $a_i \leq b_i$ for every $i \in [n]$. (For example $(1, 0, 1) \leq (1, 2, 2)$, while $(1, 0, 1)$ and $(0, 1, 2)$ are incomparable.)

For a sequence $a = (a_1, a_2, \ldots, a_n)$ define the weight of $a$ to be $w(a) := a_1 + a_2 + \ldots + a_n$. A chain $C = (a_1, a_2, \ldots, a_k)$ with $a_1 < a_2 < \ldots < a_k$ in $T_n$ is called symmetric if $w(a_{i+1}) = w(a_i) + 1$ for $i = 1, 2, \ldots, k - 1$ and $w(a_1) + w(a_k) = 2n$.

a) Show that $T_n$ allows a symmetric chain decomposition.

b) Give an example of an antichain in $T_n$ which intersects every symmetric chain. Deduce that this antichain is maximum.

3. **Hilton, 1974.** Let $1 \leq g \leq h \leq n$ be integers with $g + h \leq n$. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be an intersecting family and suppose that $g \leq |F| \leq h$ for every $F \in \mathcal{F}$. Use Erdős-Ko-Rado theorem to show that

$$|\mathcal{F}| \leq \sum_{i=g}^{h} \binom{n-1}{i-1}.$$ 

4. Let $1 \leq r \leq n/2$. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be an intersecting Sperner family. Suppose that $|A| \leq r$ for every $A \in \mathcal{F}$. Show that

$$|\mathcal{F}| \leq \binom{n-1}{r-1}.$$
5. A \textit{k-sunflower} in a set system \( \mathcal{F} \) on \( X \) is a collection of distinct sets \( F_1, F_2, \ldots, F_k \in \mathcal{F} \) such that for some \( Z \subseteq X \) we have \( F_i \cap F_j = Z \) for all \( 1 \leq i < j \leq k \). (I.e. the intersection of every pair of distinct sets in the sunflower is the same.) Let \( c(k, r) \) denote the maximum possible size of a set system \( \mathcal{F} \) such that

\[(*) \quad |F| \leq r \text{ for every } F \in \mathcal{F}, \text{ and } \mathcal{F} \text{ does not contain a } k\text{-sunflower.}\]

Suppose that a set system \( \mathcal{F} \) on \( X \) satisfies (*)..

\textbf{a)} Show that there exists a set \( Y \subseteq X \) with \( |Y| \leq (k-1)r \) such that every set in \( \mathcal{F} \) contains an element of \( Y \).

\textbf{b)} Let \( \mathcal{F}_y = \{F - y | F \in \mathcal{F}, y \in F\} \). Show that \( |\mathcal{F}_y| \leq c(k, r - 1) \) for every \( y \).

\textbf{c)} Deduce from a) and b) that

\[c(k, r) \leq (k - 1)^r r!\]

\textbf{d)} Construct an explicit example of a family \( \mathcal{F} \) satisfying (*) to show that

\[c(k, r) \geq (k - 1)^r.\]

6. Let \( r \geq 1 \) be an integer, \( A \subseteq X^{(r)} \) and \( i, j \in X \). Show that

\[|\partial \tilde{R}_{ij}(A)| \leq |\partial A|.\]