Assignment # 2: Turán- and Ramsey-type problems.

Due in class on Thursday, March 29th.

1. Let $G$ be a graph on $n$ vertices for some $n \geq 3$ with $|G| \geq \left\lfloor \frac{n^2}{4} \right\rfloor + 1$.
   
   a) Show that $G$ contains at least $\left\lfloor \frac{n^2}{4} \right\rfloor$ triangles.
   
   b) Show that the bound in a) is tight: For every $n \geq 3$ there exists a graph $G$ on $n$ vertices with $|G| = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$ containing exactly $\left\lfloor \frac{n^2}{4} \right\rfloor$ triangles.

2. Let $K_{s,s,s}$ denote the 3-graph, whose vertices can be partitioned into three sets $A_1, A_2$ and $A_3$, such that $|A_i| = s$ for $i = 1, 2, 3$, and the edges are all the triples $\{x_1, x_2, x_3\}$ such that $x_i \in S_i$ for $i = 1, 2, 3$. Show that $\pi(K_{s,s,s}) = 0$ for every $s$.

3. Bollobás. 8.7. Let $K_4^{(3)}$ denote the complete 3-graph on 4 vertices, i.e. the 3-graph isomorphic to $[4]^{(3)}$. Following de Caen (1983), we give an upper bound on $\pi(K_4^{(3)})$. Let $F \subseteq [n]^{(3)}$ be a hypergraph containing no $K_4^{(3)}$ with $|F| = m$. For $x, y \in [n], x \neq y$ let
   
   $$A(x, y) := \{z \in [n] | \{x, y, z\} \in F\},$$
   
   and let $a_{xy} := |A(x, y)|$. Note that if $\{x, y, z\} \in F$ then $A(x, y) \cap A(y, z) \cap A(z, x) = \emptyset$ and so
   
   $$a_{xy} + a_{yz} + a_{zx} \leq 2n - 3.$$

   Summing over all edges of $F$ deduce that
   
   $$\sum_{\{x,y\} \in [n]^{(2)}} a_{xy}^2 \leq (2n - 3)m.$$

   Using convexity of $x^2$ show that the left hand side is at least $(3m)^2/(\binom{n}{2})$ and deduce that $m \leq \frac{2n-3}{9} \binom{n}{2}$ and $\pi(K_4^{(3)}) \leq 2/3$.

4. Let $G$ be a graph with $V(G) = [17]$ and $x, y \in V(G)$ adjacent if and only if
   
   $$(x - y) \mod 17 \in \{\pm 1, \pm 2, \pm 4, \pm 8\}.$$

   a) Show that neither $G$ nor the complement of $G$ contains a $K_4$ subgraph.
   
   b) Deduce that $R(4, 4) = 18$.

5. Schur’s theorem. Show that for every positive integer $k$ there exists a positive integer $n$ satisfying the following. In every coloring of $[n]$ with $k$ colors it is possible to find a triple of (not necessarily distinct) integers $x, y, z$ of the same color so that $x + y = z$.
   
   (Hint: Use Ramsey’s theorem.)

6. Show that for each $\varepsilon > 0$ there exists $N$ with the following property. For each real $\alpha > 0$ there exist integers $q$ and $p$ such that $1 \leq q \leq N$ and
   
   $$|q^2 \alpha - p| \leq \varepsilon.$$

   (Hint: Use van der Waerden’s theorem.)