Lecture 1: Introduction
The probabilistic method -
- to find an object with certain properties
  choose a random one (from carefully
  selected distribution).

Pioneered by Paul Erdős.

Lemma 1.1: Every graph with \( m \) edges has
  a bipartite subgraph with \( \geq \frac{m}{2} \) edges.
  (Graphs are simple:
  Graph \( G \) consists of vertex set \( V(G) \)
  and edge set \( E(G) \subseteq V(G) \times V(G) \)
  collection of vertex pairs).

Proof: Let \( (A, B) \) be a partition of \( V(G) \)
  chosen uniformly at random.
  \( P[\{v \in A\}] = \frac{1}{2} \) independently for every \( v \in V(G) \).

Let \( E' \subseteq E(G) \) consists of edges with one end in \( A \) another in \( B \).

\[
P[ee E'] = \frac{1}{2}.
\]

\[
E[|E'|] = \sum_{ee E'} P[ee E'] = \frac{1}{2} |E'| = \frac{1}{2} m.
\]
Ramsey number is the minimum integer $N$ s.t. in every coloring of edges of complete graph on $N$ vertices in two colors red & blue, there is either a complete subgraph on $k$ vertices with all red edges or $L$ vertices with all blue edges.

**Ramsey theorem:** $R(k, l)$ exists, for all $k, l \geq 1$.

**Erdős–Szekeres:** $R(k, k) \leq \binom{2k-2}{k-1} \sim c \frac{4}{\sqrt{k}}$.

**Theorem 1.2:** If \( \left( \binom{n}{k} \right)^{2^{-\binom{2}{2}}} < 1 \) (\( \star \)), then $R(k, k) \geq n$.

**Proof:** Color the edges of $K_n$ in red or blue uniformly independently at random. (there are $2^{\binom{n}{2}}$ colorings, this is the same as choosing one at random).

Assume $V(K_n) = \{1, 2, \ldots, n\}$, $\left| V(K_n) \right| = n$.

For every $X \subseteq \{1, 2, \ldots, n\}$ with $|X| = k$
let $A_x = \begin{cases} 1 & \text{if all edges between vertices of } X \\ 0 & \text{otherwise} \end{cases}$

$$P[A_x = 1] = \frac{1}{2^{(k-1)}} = 2 \cdot \left(\frac{1}{2}\right)^{(k)}$$

$$E[\Sigma A_x] = \sum_x E[A_x] = \frac{1}{2^{(k-1)}} \cdot \binom{\frac{n}{k}}{(x)} \leq 1.$$

So $P[\Sigma A_x = 0] > 0 \rightarrow$ there exists a coloring with no monochromatic $K_k$.

$$R(k,k) \geq 2^{\frac{k}{2}}$$

$$R(k,k) \geq (1 + o(1)) \frac{k}{\sqrt{2}} 2^{\frac{k}{2}}$$

The best explicit constructions only give colorings with no monochromatic $K_k$ on $\sim 2^{\frac{k}{2}}$ vertices.
Lemma 1.3: For all $k, n \in \mathbb{N}$

$$R(k, k) > n - \binom{n}{k} 2^{-\left(\frac{k}{3}\right)} \quad (***)$$

"Alteration method"

Proof: As seen in 1.2, there is a $2$-coloring of $K_n$ with $\leq \binom{n}{k} 2^{-\left(\frac{k}{3}\right)}$ monochromatic $K_k$.
Remove a vertex from each of them to obtain a graph coloring of complete graph with $\geq (***)$ vertices and no monochromatic $K_k$.

$$\downarrow$$

$$R(k, k) \geq (1+o(1)) \frac{k}{e} 2^{k/2}.$$ 

Best known bound:

$$R(k, k) \geq c \frac{k^2}{2^{k/2}}.$$ 

Upper bound:

$$R(k, k) \leq e^{-(\log k)^2} \binom{2k}{k}$$

Ashwin Sah 2020
**k-uniform hypergraphs**

H has vertex set \( V(H) \)
and edge set \( E(H) \subseteq V(H)^{(k)} \)

collection of \( k \) element subsets of \( V(H) \).

H is \( 2 \)-colorable if there exists a 2-coloring of \( V(H) \)
s.t. every edge contains two vertices of different colors.

What is the minimum number of edges in a non \( 2 \)-colorable \( k \)-uniform hypergraph?

Let \( m(k) \) denote the answer.

\[ m(2) = 3 \quad m(3) = 7 \]

\( K_{5,3} \) all triples

\( K_{5} \) on 5 vertices.

10 edges

Fano plane 3-uniform hypergraph.

7 edges
Lemma 1.4: \( m(k) \geq 2^{k-1} \) for all \( k \geq 2 \).

Proof: If \( H \) is \( k \)-uniform \( |E(H)| \leq 2^{k-1} \) we want to show there is a \( 2 \)-coloring.

Color each vertex white or black independently uniformly at random.

\[
\Pr[ \text{e is monochromatic}] = 2 \cdot \frac{1}{2^k} = \frac{1}{2^{k-1}} \\
\mathbb{E}[\text{\# monochromatic edges}] = |E(H)| \cdot \frac{1}{2^{k-1}} < 1.
\]

So there is a coloring with no monochromatic edges.

\( m(k) \neq O(k^2 2^k) \)