Lecture 14: Janson inequalities

\[ P(\bar{A}_1 \land \ldots \land \bar{A}_n) \leq e^{-\mu + \frac{\sigma}{2}}. \]
Recall: The FKG inequality:

\[ f, g : \mathcal{P}(\mathcal{N}) \to \mathbb{R}^+ \text{ increasing} \]

then \( E_\mu[fg] \geq E_\mu[f] \cdot E_\mu[g] \)

for any log-supermodular prob. distribution \( \mu \) on \( \mathcal{P}(\mathcal{N}) \).

In particular, if \( A, B \subseteq \mathcal{P}(\mathcal{N}) \) are increasing families

and \( \mu = [\mathcal{N}]_p \), then

\[ \Pr(\text{any } R \in A \cap B) \geq \Pr(\text{any } R \in A) \cdot \Pr(\text{any } R \in B) \]

for \( R \) selected randomly according to \( [\mathcal{N}]_p \).

Remark:

- FKG also applies to any distribution on \( \mathcal{P}(\mathcal{N}) \)
  where elements are selected independently but possibly with different probabilities.

- \( \Pr(\text{any } R \in A \cup A_1 \cup \cdots \cup A_k) \geq \prod_{i=1}^k \Pr(\text{any } R \in A_i) \)

  whenever \( A_i \subseteq \mathcal{P}(\mathcal{N}) \) are increasing.
8. Janson inequalities.

What is the probability that $G(n, p)$ contains no $K_3$ subgraph?

Setup of the results in this section

Let $R \in \mathcal{P}([N])$ selected by independently selecting its elements

Let $S_1, S_2, \ldots, S_k \subseteq [N]

We want to upper bound the probability that $R$ contains none of $S_i$. We have $N = \binom{n}{2}, |S_i| = 3, k = \binom{n}{3}$,

$S_i$ correspond to $\Delta$'s in $K_n$.

Let $A_i$ be the event \{ $S_i \subseteq R$ \}. Let $X = \sum_{i \in \mathbb{E}} 1_{A_i}$, i.e., $X$ is the number of events $A_i$ that occur, $X = \# \text{ of sets } S_i \text{ that are in } R$.

We are interested in $P(X = 0) = P(R \notin A_1 \cup \overline{A}_2 \cup \ldots \cup \overline{A}_k)$.

Let $\mu = \mathbb{E}(X) = P(R \in \bigcap \overline{A}_i \cap \overline{A}_2 \cap \ldots \cap \overline{A}_k)$.
\[ P(X = 0) = P(\bar{A}_1 \land \bar{A}_2 \land \ldots \land \bar{A}_k) \geq \prod_{i=1}^{k} P(\bar{A}_i) = \prod_{i=1}^{k} (1 - P(A_i)) \quad P(A_i) = o(1) \]

\[ \prod_{i=1}^{k} P(A_i) - (1 - o(1)) \leq P(A_i) \]

\[ \prod_{i=1}^{k} e^{P(A_i) - (1 - o(1))} \leq \prod_{i=1}^{k} e^{P(A_i)} \]

**Lemma 8.1:** IF \[ P(A_i) = o(1) \] then \[ P(X = 0) \geq e^{-\frac{c_1}{1 - o(1)}} \]

Let \[ \Delta = \sum_{i,j} P(A_i \land \bar{A}_j) \]

where \( i \neq j \) if \( S_i \cap S_j \neq \emptyset \)

Note that \( A_i \) is independent of \( \{ A_j \}_{j \neq i} \)

Recall that \( \Delta \) appeared in Chebyshev's inequality \( P(X = 0) = o(1) \), which implied that if \( \mu \to \infty \) \( \Delta = o(\mu^2) \) then
Theorem 8.2: In the setting above

First

Janson inequality

Combining 8.1 & 8.2, we get that if \( P(A_i) = o(1) \) \( \implies \Delta = O(\mu) \)

then \( P(X = 0) = e^{-(1-o(1))\mu} \)

For the problem of estimating prob. that \( G(n,\mu) \) has no \( K_3 \).

\( A_1, A_2, \ldots, A_{\binom{n}{3}} \) each corresponding to presence of a particular \( \Delta \) in \( G(n,\mu) \)

\[ M = p^3 \binom{n}{3} = \frac{p^3 n^3}{6} \]

\[ \Delta = \Theta(n^3 p^5) \quad n^4 p^5 < \Delta \leq o(1) \quad n p^2 \leq o(1) \]

So \( \Delta = o(\mu) \) when \( p = o\left(\frac{1}{\sqrt{n}}\right) \).

Corollary 8.3: For \( p = o\left(\frac{1}{\sqrt{n}}\right) \)

\[ P(G(n,\mu) \text{ has no } K_3) = e^{-\frac{p^3 n^3}{6}} \]

If \( p = \frac{c}{n} \)

\[ P(G(n, \frac{c}{n}) \text{ has no } K_3) = e^{-\frac{c^3}{6}} \]
Proof:

Let $r_i = P(A_i | \bar{A}_1 \land \bar{A}_2 \land \ldots \land \bar{A}_{i-1})$

$P(x = 0) = P(\bar{A}_1 \land \bar{A}_2 \land \ldots \land \bar{A}_k) = P(\bar{A}_1)P(\bar{A}_2 | \bar{A}_1)P(\bar{A}_3 | \bar{A}_2 \land \bar{A}_1) \cdots P(\bar{A}_k | \bar{A}_{k-1})$

$= (1-r_1)(1-r_2) \ldots (1-r_k)$

$\leq e^{-r_1 - r_2 - \ldots - r_k}$

Need to show $\sum r_i \geq M - \frac{\Delta}{2} = \sum_i P(A_i) - \sum_{i,j \neq i} P(A_i \land A_j)$

We will show $r_i \geq P(A_i) - \sum_{j \neq i} P(A_i \land A_j)$

It implies the theorem.
Let \( D_0 = \bigwedge_{j<i} \overline{A_j}, \quad D_i = \bigwedge_{j<i} \overline{A_j} \). \( D_i \) is decreasing.

Then \( r_i = \frac{\Pr(A_i \land D_0 \land D_i)}{\Pr(D_0 \land D_i)} = \frac{\Pr(A_i \land D_0, D_i)}{\Pr(D_0 \land D_i)} \)

\[
\geq \frac{\Pr(A_i \land D_0, D_i)}{\Pr(D_0)} \quad \text{increasing} \quad \frac{\Pr(A_i \land D_0)}{\Pr(D_0)} \quad \text{increasing}
\]

\[
= \Pr(A_i \land D_0) - \Pr(A_i \land D_0, D_i)
\]

\[
= \Pr(A_i \land D_0) - \Pr(A_i \land D_0, D_i)
\]

\[
\Pr(A_i) = \Pr(A_i \land D_0)
\]

\[
\Pr(A_i) = \Pr(A_i \land D_0)
\]

Remains to check that

\[
\Pr(A_i \land \overline{D}_0) \leq \sum_{j<i} \Pr(A_i \land A_j)
\]

\[
\Pr(A_i \land (\bigvee A_j)) \leq \sum_{j<i} \Pr(A_i \land A_j) \quad \text{union bound}
\]

\[
\bigvee_{j<i} (A_i \land A_j)
\]
What about setting when $\mathcal{E}(M \leq \Delta)$ can we give meaningful upper bounds on $P(X=0)$.

**Theorem 8.3**: In our setting if $\Delta \geq \mu$

\[ P(X = 0) \leq e^{-\frac{\mu^2}{2\Delta}} \]

Second Janson inequality

The proof parallels the bootstrapping method used in crossing lemma.

Select random subset of events with certain probability & apply 8.2.

- This will allow us to estimate the prob $G(n,p)$ is $\Delta$-free for remaining regime $p = \Omega\left(\frac{1}{\log n}\right)$

and to prove that

\[ X(G(n, \frac{1}{2})) = (1 + o(1)) \frac{n}{2 \log n} \]