Lecture 15: Second Janson Inequality & Applications

\[ \ll \frac{n}{\log_2 n} \approx 2 \log_2 n \approx 2 \log_2 n \approx 2 \log_2 n \]
Setup: We select random subset $R$ of $\mathbb{N}$ by selecting its elements independently (possibly with different probabilities).

$s_i \in [N]$  $A_i$ - event that $s_i \in R$

$X = \sum 1_{A_i}$ \# of $s_i$'s in $R$. \quad \mu = E[X]$

$\Delta = \sum_{i \neq j} P(A_i \land A_j)$ \quad (i.e. $s_i \cap s_j \neq \emptyset$)

8.2. $P(X=0) \leq e^{-\mu + \frac{\Delta}{2}}$  First Janson inequality.

First Janson inequality. Works in the regime $\mu \geq \Delta$.

$P(X=0) \geq e^{-(1-o(1))\mu}$ by FKG inequality if $P(A_i) = o(1)$ for all $i$.

Theorem 8.3: In the setting above if $\Delta \geq \mu$

Second Janson inequality

$P(X=0) \leq e^{-\mu^2/2\Delta}$

Proof: By bootstrapping First Janson inequality.
Let $S_1, S_2, \ldots, S_k$ be the sets in the setup and $A_1, A_2, \ldots, A_k$ corresponding events.

Select $I \subseteq [k]$ by choosing each $i \in [k]$ independently with probability $q$ (to be chosen).

Apply 8.2. to sets $\mathcal{S}_I \mathcal{S}_{i \in I}$.

\[ P(X = 0) \leq P(X_I = 0) \leq e^{-M_I + \frac{\Delta_I}{2}}. \]

For fixed $I$: $M_I = \mathbb{E} \left[ \sum_{i \in I} 1_{A_i} \right]$ and $M = \mathbb{E} \left[ \sum_{i \in [k]} 1_{A_i} \right]$.

\[ \Delta_I = \sum_{i \neq j} P(A_i \land A_j) \quad \Delta = \sum_{i \neq j} P(A_i \land A_j). \]

\[ q = \frac{\Delta}{\Delta} \]

\[ \mathbb{E} \left[ -M_I + \frac{\Delta_I}{2} \right] = -qM + \frac{q^2}{2} \Delta = -\frac{\mu^2}{2\Delta} \]

\[ \mathbb{E} [M_{\varnothing}] = qM, \quad \mathbb{E} [\Delta_I] = q^2 \Delta \]

\[ P(X = 0) \leq e^{-\frac{\mu^2}{2\Delta}} \quad \checkmark. \]
8.28 8.3

\[ P(X = 0) = \begin{cases} 
 e^{-\frac{\mu^2}{2\Delta}} & \text{if } \mu \leq \Delta \\
 0 & \text{otherwise}
\end{cases} \]

Probability that \( G(n, p) \) has no \( K_3 \) subgraph (triangle-free).

We already proved that for \( p = o\left(\frac{n^{-1/2}}{}\right) \)

\[ P(G(n, p) \text{ is triangle-free}) = e^{-\frac{n^3p^3}{6}} \]

By 8.3.

\[ P(G(n, p) \text{ is triangle free}) \leq e^{-\Omega(n^2p)} \]

Is there a matching lower bound.

\[ P(G(n, p) \text{ is edgeless}) = (1 - p)^\binom{n}{2} = e^{-\Theta(p)n^2} \geq e^{-\Omega(n^2p)} \]

Theorem 8.4: For \( p < 0.99 \)

\[ P(G(n, p) \text{ is triangle free}) = \begin{cases} 
 e^{-\Theta(n^3p^3)} & \text{if } p < n^{-1/2} \\
 e^{-\Theta(n^2p)} & \text{if } p > n^{-1/2}
\end{cases} \]
For $p = \frac{1}{2}$, $G(n, \frac{1}{2})$ uniformly samples graphs $G$ with $V(G) = [n]$.

So estimating $P(G(n, \frac{1}{2}) \text{ is triangle free})$ is estimating the number of triangle-free graphs.

Calculating the number of triangle-free graphs $\leq 2^{\Theta(\frac{n^2}{2})}$.

Total number of graphs $2^{\binom{n}{2}}$.

Erdős, Kleitman & Rothschild 1976:

Almost all triangle-free graphs are bipartite $\sim \Theta(n^2)$.

There are $2^{\Theta(n^2)}$ triangle-free graphs.

$P(G(n, \frac{1}{2}) \text{ is triangle-free}) = 2^{\Theta(-n^2 + 3)}$.

In fact, the same logic applies for $p \geq n^{-\frac{1}{2} + \epsilon}$. 
Chromatic number of $G(n, \frac{1}{2})$.

Recall: The clique number $\omega(G)$ of $G$ is the size of the largest complete subgraph.

Behaviour of $\omega(G(n, \frac{1}{2}))$.

Estimate probability $\omega(G(n, \frac{1}{2})) < k$.

To set up things so that Janson inequalities are applicable let $S_i$ correspond to edge set of complete subgraphs on $k$ vertices in $G(n, \frac{1}{2})$.

$$M = M(k) = \binom{n}{k} \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

If $\Delta = O(\mu^2)$ and $\mu(k) \to \infty$ then $w(G(n, \frac{1}{2})) \leq k$ w.h.p.

If $\Delta = O(\mu^3)$ and $\mu(k) \to \infty$ then $w(G(n, \frac{1}{2})) \leq 2k$ w.h.p.

$$\frac{\mu(k+1)}{\mu(k)} = 1 + O\left(\frac{1}{k}\right)$$

for $k = (1 - o(1))n$.

$$\frac{\mu(k)}{\mu(k+1)} = n \left(1 - o(1)\right)$$

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for $k = (1 - o(1))n$.
Theorem 8.5: Let $k_0$ be maximum s.t. $(k_0$ depends on $n$)

$$m(k_0) = \binom{n}{k_0} \left(\frac{1}{2}\right)^{\frac{k_0}{2}} \geq 1.$$ 

Then

$$P \left( \omega(G(n, \frac{n}{2})) < k_0 - 3 \right) \leq e^{-n} \quad o(1).$$

Let $k = k_0 - 3$

Proof:

By calculations above $3 - o(1)$

$$m(k) \geq n.$$

Let us estimate $\Delta(k)$

Intuition (details can be routinely checked)

the highest order term comes from pairs

$S_i S_j$ where corresponding cliques have

2 vertices in common.

\[
\sum_i P(A_i \land A_j) - \sum_i P(A_i) \cdot \sum_j P(A_j) = M^2
\]

\[
2 \cdot \left(\frac{k}{n}\right) \cdot \left(\frac{k}{n}\right)^2
\]

\[
\sim \text{proportion of pairs that have 2 vertices in common}
\]

We have

$$\Delta(k) \Theta = \Theta \left(\frac{k^4}{n^2} M^2\right) = o(M^2).$$

\[\Rightarrow \quad M > M.\]
By 2nd Janson's inequality

\[
\Pr(\omega(G(n, \frac{1}{2})) < k) \leq \Pr(X=0) \leq e^{-\frac{M^2}{2\Delta}}
\]

\[
= e^{-\Theta\left(\frac{n^2}{\ln n}\right)} = e^{-n^2 \cdot o(1)}
\]

\[
\Rightarrow k \sim 2 \log_2 n
\]

Let \(\alpha(G)\) denote the independence number of \(G\) - the maximum size of set of pairwise non-adjacent vertices.

By 8.5

\[
\Pr(\alpha(G(n, \frac{1}{2})) < k_0 - 3) \leq e^{-n^2 \cdot o(1)}
\]

**Theorem 8.6:** With high probability,

\[
\chi(G(n, \frac{1}{2})) = (1 + o(1)) \frac{n}{2 \log_2 n}
\]

**Proof:**

\[
\chi(G(n, \frac{1}{2})) \geq (1 - o(1)) \frac{n}{2 \log_2 n}
\]

with high probability by Markov

\[
\Pr(\alpha(G(n, \frac{1}{2})) \leq k_0 + 1) = (1 + o(1)) \frac{n}{2 \log_2 n}
\]

\[
k_0 = (1 + o(1)) \frac{n}{2 \log_2 n}
\]

But

\[
\chi(G) \geq \frac{|V(G)|}{\alpha(G)}
\]

For any \(G\), implying
We will show that for any
\[
m \geq \left\lceil \frac{n}{\log^2 n} \right\rceilor every subset \( X \subseteq [n] \)

with high probability
\[
d \left( G[X] \right) \geq (1 - o(1))2 \log n.
\]

subgraph induced by \( X \)
For every \( X \subseteq [n] \), \(|X| \geq m \).
where \( G = G(n, 1/2) \).

This implies the theorem as we can "pick out" color classes of size \( n \cdot 2 \log n \). until \( \leq m \) vertices are left.

implying
\[
X(G) \leq (1 + o(1)) \frac{n}{2 \log n} + m = (1 + o(1)) \frac{n}{2 \log n}.
\]

Our claim follows from 8.5 (for \( d \) instead of \( w \)).

By union bound
\[
\Pr \left[ d \left( G[X] \right) < (1 - o(1))2 \log n \right] \leq e^{-n^{2 - o(1)}}
\]
for each such \( X \).

there are \( \leq 2^n \) choices of \( X \).