Lecture 4: Linearity of Expectation: Crossings, Vectors & Cycles.
Recall:

**Theorem 2.5:** Let $G$ be a graph with $n$ vertices & $m$ edges, $m \geq 4n^3/n^2$.

Then $\text{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}$

**Proof:** From Euler's Formula $m - 3n$

$\text{cr}(H) \geq |E(H)| - 3|V(H)|$.

Select $X \subseteq V(G)$ by choosing each vertex independently with probability $p$.

Let $H$ be a random subgraph of $G$ with $V(H) = X$.

$E[H] \subseteq E(G)$ consists of all edges with both ends in $X$.

$(H$ is induced by $X)$.

$E[\text{cr}(H)] \geq E[|E(H)|] - 3E[|V(H)|]$.

$\sum_{x \in V(G)} p(x \in X)pn$
\[ p^4 \cr(G) \geq p^2 m - 3pn \]

\[ \cr(G) \geq mp^2 - 3np^{-3} \quad \rightarrow \text{choose } p \text{ to maximize} \]

\[ \frac{\partial}{\partial p} = -2mp^{-3} + 9np^{-4} \times p^4 \]

\[ p = \frac{9n}{2m} \]

\[ \cr(G) \geq m \left( \frac{m}{4n} \right)^2 - 3n \left( \frac{m}{4n} \right)^3 = \left( \frac{1}{16} - \frac{3}{64} \right) \frac{m^3}{n^2} = \frac{1}{64} \frac{m^3}{n^2} \]

Crossing Lemma has many nice unexpected applications.

see notes for MATH 550,

Yufei Zhao's notes.
(i.e. sum-product inequalities)
Sums of vectors

\[ H_n = \{ x_1, x_2, \ldots, x_n \mid x_i \in \{0, 1\} \} \]

- n-dimensional hypercube
- \( \{ x_1, x_2, \ldots, x_n \mid x_i \in \{0, 1\} \} \)

\[ V_I = \sum_{i \in I} V_i \]

- radius of \( H_n = \sqrt{n \cdot (\frac{1}{2})^2} = \sqrt{\frac{n}{2}} \)
- Balls of radius \( \sqrt{\frac{n}{2}} \) centered at vertices cover \( H_n \)
- Interior of \( H_n \)

Now relax the assumption that \( V_1, V_2, \ldots, V_n \) is an orthonormal basis \( \mathcal{B} \), only assume for every \( i \)

\[ |V_i| = 1 \]

\[ \mathcal{S} = \{ V_I \mid I \subseteq [n] \} \]

- maximum radius of \( S = \frac{n}{2} \) (\( \frac{n}{2} \) always suffices, exercise)

What about covering half of \( S \)?

- \( V_i = (1, 0, \ldots, 0) \)

- \( (\frac{n}{2} - 1) + (\frac{n}{2} - 1) + (\frac{n}{2}) + (\frac{n}{2} + 1) \ldots + (\frac{n}{2} + 1) \)

- \( \frac{n}{2} \)
Theorem 2.6: Let \( v_1, v_2, \ldots, v_n \in \mathbb{R}^n \) s.t. \( |v_i| = 1 \).

Let \( S = \{ v_I \mid I \subseteq [n] \} \) (considered with multiplicities).

Then \( |v_I - w| \leq \sqrt{n}/2 \) for at least \( 2^{n-1} \) \( I \subseteq [n] \) where

\[
    w = \frac{v_1 + v_2 + \ldots + v_n}{2}.
\]

Markov's inequality

Let \( X \geq 0 \) random variable. Then

\[
P[X \geq a] \leq \frac{E[X]}{a}
\]

for every \( a > 0 \).

Proof:

\[
    E[X] = \sum_{0}^{a} \underbrace{E[X | X < a]}_{\text{v}} P[X < a] + \underbrace{E[X | X \geq a]}_{\text{v}} P[X \geq a]
\]

\[
    \geq a P[X \geq a].
\]

\[
P[X \leq a] \geq 1 - \frac{E[X]}{a}.
\]

Example: \( a = 2E[X] \)

\[
P[X \leq 2E[X]] \geq \frac{1}{2}.
\]
Proof: \[ V_I - W = \sum_{i \in [n]} E_i V_i \]

So we assume that \( E_i \) are independently chosen so that \( \Pr[E_i = \frac{1}{2}] = \Pr[E_i = -\frac{1}{2}] = \frac{1}{2} \).

\[
E\left[ \sum_{i \in [n]} E_i V_i \right]^2 = E\left[ \left( \sum_{i \in [n]} E_i V_i \right) \left( \sum_{j \in [n]} E_j V_j \right) \right]
\]

\[
= \sum_{i, j \in [n]} \sum_{i \neq j} E[E_{i,j}] <v_i, v_j> = \sum_{i \in [n]} \frac{1}{n} \|V_i\|^2 = \frac{n}{n}.
\]

If \( i \neq j \) then \( E[E_{i,j}] = E[E_i] E[E_j] = 0 \).

By Markov,
\[
\Pr\left[ \sum_{i \in [n]} E_i V_i \leq \frac{n}{2} \right] \geq \frac{1}{2}.
\]

\[
\Pr\left[ \|V_I - W\| \leq \sqrt{\frac{n}{2}} \right] \geq \frac{1}{2}.
\]

\( I \) is chosen independently random.
\[ \{p_1 v_1 + p_2 v_2 + \cdots + p_n v_n \mid p_i \in [0,1] \} \]

cover it with balls centered at "vertices"

\[ V_I, I \subseteq [n] \]

How big a radius do we need? Given \(|V_I| = 1\).

What if \(q v_1 = v_2 = \cdots = v_n\)? \[ \frac{1}{2} \]

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Theorem 2.7: Let \(v_1, v_2, \ldots, v_n \in \mathbb{R}^n\) be s.t. \(|v_i| = 1\).

Then for every \(w = p_1 v_1 + p_2 v_2 + \cdots + p_n v_n\) s.t. \(p_i \in [0,1]\)

there exists \(I \subseteq [n]\) s.t.

\[ |w - V_I| \leq \frac{\sqrt{n}}{2} \]

(i.e. balls of radius \(\frac{\sqrt{n}}{2}\) centered at vertices cover interior).

Proof sketch

\[ w - V_I = \sum_{i \in [n]} \varepsilon_i V_i \]

\[ \varepsilon_i = \begin{cases} p_i & \text{if } i \not\in I \\ p_i - 1 & \text{if } i \in I \end{cases} \]

Select I at random. Let \(P[i \in I] = p_i\) for each \(i \in [n]\), independently for each \(i\).

\[ E[\varepsilon_i] = p_i (p_i - 1) + (1 - p_i) p_i = 0. \]

Calculations as in Theorem 2.6 yield the result.
Girth & chromatic number of graphs

\( \chi(G) \) the chromatic number is the minimum number of colors needed to color vertices s.t. adjacent vertices receive different colors.

\( \chi(G) \) is hard to determine.

- Showing \( \chi(G) \leq k \) is not hard \( \rightarrow \) show coloring
- What about \( \chi(G) > k \) \( \rightarrow \) Maybe we can find a small obstruction to \( k \)-coloring?
  i.e. \( K_{k+1} \) complete subgraph on \( (k+1) \) vertices.
  \[ \chi(K_{k+1}) = k+1 \].

We will show that there exist graphs with \( \chi(G) \geq k \) and every subgraph on \( \leq L \) vertices is two-colorable for any pair \( k \& L \).

Examples come from random graphs.
Erdős–Rényi random graph $G(n,p)$ is a graph with $n$ vertices where every pair of vertices is chosen to be adjacent independently with probability $p$.

Expected number of triangles in $G(n,p)$

$$\Delta \qquad \frac{n(n-1)(n-2)}{3} p^3 < 1$$

If expected number $< 1$ then $p < \frac{\Theta^2}{\Theta n}$

But then average degree $< 2$, the whole graph is likely two-colorable.

To be continued.