Lecture 8: Existence of Thresholds & Chernoff Bounds
Previously:

\( [n] \_p \) - distribution on subsets of \([n] = \{1, 2, \ldots, n\} \)

obtained by selecting each element at random with probability \( p \).

\( F \subseteq \mathcal{P}(\{n\}) \) is \textit{monotone} if for \( A \subseteq B \subseteq [n] \)

if \( A \in F \) then \( B \in F \).

\textit{Non-trivial} if \( \emptyset \notin F \), \([n] \notin F \).

Lemma 4.6: \( F \subseteq \mathcal{P}(\{n\}) \) monotone & non-trivial (for fixed \( [n] \))

then \( f(p) = \sum_{A \subseteq F} p^{|A|} (1-p)^{|n|-|A|} \)

is continuous & increasing.

Proof: \( f(p) = \sum_{A \in F} p^{|A|} (1-p)^{|n|-|A|} \rightarrow \text{continuous} \).

Increasing \( \rightarrow \) by coupling.

\( r: [n] \rightarrow [0, 1] \)

\( (r(1), r(2), \ldots, r(n)) \) each \( r(i) \) is selected independently

uniformly in \([0, 1]\).

\( [n]_p = \{ i : r(i) \leq p \} \).

\( p_1 < p_2 \) selecting \( A = \{ i : r(i) \leq p_1 \} \)

\( B = \{ i : r(i) \leq p_2 \} \)

\( p_1 < p_2 \) generating \((A, B)\) s.t.

\( A \) is distributed according to \([n]_p \)

\( p_1 < p_2 \) and \( A \subseteq B \).
\[ f(p_1) = P(A \in F) \leq P(B \in F) = f(p_2) \]

So \( f(p) \) is non-decreasing.

With probability > 0 we have \( A = \emptyset \), \( B = [n] \), \( (p_2 - p_1)^n \).

So \( f(p_1) \leq f(p_2) - (p_2 - p_1)^n \) so \( f(p) \) strictly increasing.

**Lemma 4.7:** Let \( F, f(p) \) be as in 4.6.

Then \( 1 - f(2p) \leq (1 - f(p))^2 \)

**Proof:** Let \( A, B \) be \( \emptyset \) subsets of \([n]\) independently selected according to \([n]p\).

How is \( A \cup B \) distributed? \([n]_{2p-p^2}\)

\[
P[i \in A \cup B] = 1 - P[i \notin A]P[i \notin B] = 1 - (1 - p)^2 = 2p - p^2 < 2p.
\]

\[
f(2p^2) = P(A \cup B \in F) \geq P(A \in F \text{ or } B \in F) = 1 - (1 - f(p))^2
\]

\[
f(2p) \geq 1 - (1 - f(p))^2 \quad \checkmark
\]
Corollary 4.8: Every non-trivial graph property has a threshold.

(Bollobás & Thomason) 1987

Proof: Need to show that there exists for each $n$

$P_c(n)$ s.t.

$\Pr(G(n, p) \in \mathcal{F}) \to 1$ as $p/P_c \to \infty$ (x)

$\Pr(G(n, p) \notin \mathcal{F}) \to 1$ as $p/P_c \to 0$ (***)

Fix $n$. $f(p) = \Pr(G(n, p) \in \mathcal{F})$.

There exists $P_c$ s.t. $f(P_c) = 1/2$, by 4.6.

By 4.7. $1 - f(2^k P_c) \leq (1 - f(P_c)) = \frac{1}{2^k}$

$f(2^k P_c) \geq 1 - \frac{1}{2^k} \to 1$ as $k \to \infty$.

implies (x)

is in the other direction

$1 - f(2^{-k} P_c) \geq (1 - f(P_c))^{1/k} \geq \frac{1}{2^{1/k}}$

$f(2^{-k} P_c) \leq 1 - \frac{1}{2^{1/k}} \to 0$ as $k \to \infty$

implies (***).
\[ F = G_0 \text{ has no isolated vertices.} \]
\[ (1-p)^n \sim e^{-pn} \]
\[ \text{# isolated } p \text{ vertices } \leq e^{-pn} \log n - pn \]

Threshold \[ \frac{\log n}{n} \]

If \[ p = \frac{\log n + c(n)}{n} \] and \[ c(n) \to +\infty \]
then \[ G(n, p) \in F \text{ w.h.p.} \]
and \[ c(n) \to -\infty \]
\[ G(n, p) \in /F \text{ w.h.p.} \]

(Same for connectivity)

Sharp threshold

\[ F \text{ has no sharp threshold if there exists } \epsilon > 0 \]
and \[ p_c(n) \text{ s.t.} \]
\[ P ( G(n, p) \in F ) \in [\epsilon, 1-\epsilon] \]
for \[ p \in [(1-\epsilon)p_c(n), (1+\epsilon)p_c(n)] \].
K\textsuperscript{3} containment or \textit{H} containment has no sharp threshold.

but connectivity, existence of Hamiltonian cycles, have sharp thresholds.

Theorem 4.9: If monotone property \( F \) has no sharp threshold (Friedgut 1999) then it can be approximated by a subgraph containment property.

[Another interesting topic: Thresholds vs. Expectation thresholds.]
5. **Chernoff bounds**

**Theorem 5.1:** Let $X = X_1 + X_2 + \ldots + X_n$, where $X_1, X_2, \ldots, X_n$ are independent and $X_i \in [-1, 1]$, and $E[X_i] = 0$.

$$P(X \geq \lambda \sqrt{n}) \leq e^{-\lambda^2/2}$$

**Proof:** Moment generating function for $X$

$$t \Rightarrow E[e^{tx}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[X^n]$$

$$E[e^{tx}] = \prod_{i=1}^{n} E[e^{tx_i}] \leq e^{\frac{t^2n}{2}}$$

$$e^{tx} \leq \frac{1 - x^2}{2} e^{-t} + \frac{1 + x^2}{2} e^{t} \text{ by convexity.}$$

$$E[e^{tx_i}] \leq e^{-t} E\left[\frac{1 - x_i^2}{2}\right] + e^{t} E\left[\frac{1 + x_i^2}{2}\right]$$

$$= \frac{e^{t} + e^{-t}}{2} \leq e^{t^2/2}$$

$\Downarrow$ exercise?
\[ P(X \geq \lambda \sqrt{n}) = 1 - P\left[e^{X} \geq e^{\lambda \sqrt{n}}\right] = \frac{1}{e^{\lambda \sqrt{n}}} \stackrel{\text{Markov}}{=} e^{-t \lambda \sqrt{n} + \frac{t^2 n}{2}} = e^{-\lambda^2 / 2} \]

\[ \frac{d}{dt} = -\lambda \sqrt{n} + t \sqrt{n} \]

\[ t = \frac{\lambda}{\sqrt{n}} \]

**Corollary 5.2:** Let \( X = X_1 + X_2 + \ldots + X_n \), \( \mu = E[X] \), \( X_i \)'s are independent, \( X_i \in [0, 1] \).

then \( P(X \geq \mu + \lambda \sqrt{n}) \leq e^{-\lambda^2 / 2} \).