Exponentially many perfect matchings in cubic graphs

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**Perfect matchings in bridgeless cubic graphs**

**A bridge**: an edge whose deletion disconnects the graph

**Cubic graph**: Every vertex is incident to exactly 3 edges

**Perfect matching**: A set of edges that covers all vertices exactly once.
Theorem (Petersen, 1891): Every bridgeless cubic graph has a perfect matching.

Observation (Tait, 1880): The Four Color Theorem is equivalent to the following:

The edge set of every planar cubic bridgeless graph is the union of three perfect matchings.

Conjecture (Berge, Fulkerson, 1971): In every bridgeless cubic graph there exists a collection of perfect matchings covering every edge exactly twice.
The number of perfect matchings

$m(G)$: The number of perfect matchings in a graph $G$

- $m(G)$ is hard to compute (Valiant, 1979)
- $m(G)$ is equal to the permanent of the graph biadjacency matrix when $G$ is bipartite
- $m(G)$ is related to meaningful chemical and physical properties of molecules represented by $G$
Perfect matchings in bridgeless cubic graphs

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**Theorem**: There exists a constant $\varepsilon > 0$ such that $m(G) \geq 2^{\varepsilon|V(G)|}$ in every cubic bridgeless graph $G$. ($\varepsilon = 1/3656$.)

Perfect matchings in bridgeless cubic graphs

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**Theorem:** There exists a constant \( \varepsilon > 0 \) such that \( m(G) \geq 2^{\varepsilon|V(G)|} \) in every cubic bridgeless graph \( G \). (\( \varepsilon = 1/3656 \).)

Conjectured by Lovász and Plummer (1970’s).

**Previous results:**

Voorhoeve (1979): \( m(G) \geq \left( \frac{4}{3} \right)^{|V(G)|/2} \) for bipartite \( G \).

Chudnovsky, Seymour (2008): \( m(G) \geq 2^{\varepsilon|V(G)|} \) for planar \( G \). (\( \varepsilon = 1/655978752 \).)

Edmonds, Lovász, Pulleyblank (1982): \( m(G) \geq n/4 + 2 (|V(G)| = n) \)

Král’, Sereni, Stiebitz (2008): \( m(G) \geq n/2 \)

Esperet, Král’, Škoda, Škrekovski (2009): \( m(G) \geq 3n/4 - 10 \)

Esperet, Kardoš, Král’ (2010): \( m(G) \) is superlinear.
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We can not say the same for $m(G)$. 

$$m^*(G) \geq m^*(G_1) \cdot m^*(G_2),$$
**Voorhoeve’s splitting trick**

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**Theorem (Voorhoeve):**

$m^*(G) \geq \left( \frac{4}{3} \right)^{|V(G)|/2 - 3}$ for every bipartite cubic graph $G$.

**Proof:**

![Diagram of graph $G_1$ with edges labeled $e$, $f_1$, $f_2$, $f_3$, $f_4$.]
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Diagram: A graph $G_1$ with labeled edges. The edges are labeled $e$, $f_1$, $f_2$, $f_3$, and $f_4$.
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![Graph notations]
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![Graph Diagram](image)

- $G_1$
- $f_1$, $f_2$, $f_3$, $f_4$
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**Theorem (Voorhoeve):** \( m^*(G) \geq \left( \frac{4}{3} \right)^{|V(G)|/2} - 3 \) for every bipartite cubic graph \( G \).

**Proof:**

\[
3m^*(G) \geq m^*(G_1) + m^*(G_2) + m^*(G_3) + m^*(G_4)
\]
Cubic bridgeless graph with $m^*(G) = 1$
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Cubic bridgeless graph with $m^*(G)=1$

Two perfect matchings $M_1$ and $M_2$ such that $M_1 \setminus M_2$ contains at least $\varepsilon |V(G)|$ disjoint cycles.
Two perfect matchings $M_1$ and $M_2$ such that $M_1 \cup M_2$ contains at least $\varepsilon |V(G)|$ disjoint cycles.
A strengthening

**Theorem**: There exists a constant $\epsilon > 0$ such that for every cubic bridgeless graph $G$ either

- $m^*(G) \geq 2^{\epsilon|V(G)|}$ or
- for some two perfect matchings $M_1$ and $M_2$ in $G$ the edge set $M_1 \cup M_2$ contains at least $\epsilon|V(G)|$ disjoint cycles.
The perfect matching polytope

With a perfect matching $M$ we associate a vector $\chi_M \in R^{E(G)}$: $\chi_M(e) = \begin{cases} 1, & e \in M \\ 0, & e \notin M \end{cases}$

The perfect matching polytope $PMP(G)$ is the convex hull of characteristic vectors of perfect matchings of $G$.

Let $\delta(X)$ denote the set of edges in the cut separating $X$ from $V(G)-X$.

**Theorem (Edmonds):** We have $w \in PMP(G)$ if and only if

- $0 \leq w(e) \leq 1$ for every $e \in E(G)$,
- $w(\delta(v)) = 1$ for every $v \in V(G)$,
- $w(\delta(X)) \geq 1$ for every odd $X \subseteq V(G)$.

A vector $w \in PMP(G)$ corresponds to a probabilistic distribution on the set of perfect matchings of $G$ such that

$$\Pr[e \in M_w] = w(e).$$

If $G$ is cubic and bridgeless then $w \equiv 1/3 \in PMP(G)$. 
Burls

A set $X \subseteq V(G)$ is $M$-alternating for a perfect matching $M$ of $G$ if there exists another perfect matching $M'$ such that $M$ only differs from $M'$ on $X$.

A set $X \subseteq V(G)$ is a burl if for every probabilistic distribution $M_w$ such that

$$\Pr[e \in M_w] = \frac{1}{3},$$

we have

$$\Pr[X \text{ is } M_w \text{-alternating}] \geq \frac{1}{3}.$$

A foliage in $G$ is a collection of pairwise disjoint burls.
A foliage
A foliage
A foliage
A foliage
A **foliage** in $G$ is a collection of pairwise disjoint burls. Let $f(G)$ denote the maximum size of a foliage in $G$.

**Lemma:** $m(G) \geq 2^{f(G)/3}$

**Proof:** Given a foliage $\{X_1, X_2, \ldots, X_k\}$ there exists $w \in PMP(G)$ such that each $X_i$ is $w$-alternating. Then

$$\Pr[\text{X_i is M_w - alternating}] \geq 1/3.$$

$$E[|\{i : X_i \text{ is M_w - alternating}\}] \geq k/3.$$

A perfect matching achieving the expected value can be independently changed to another perfect matching on each of the $k/3$ disjoint burls.
**Examples of burls: Twigs**

**Lemma**: For a cubic bridgeless graph $G$,

- $m^*(G) \geq 1$,
- $m^*(G) \geq 2$, if $|V(G)| \geq 6$ and $G$ has no non-trivial cuts of size $\leq 3$,
- $m(G) \geq 4$, if $|V(G)| \geq 6$. 
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A set $X \subseteq V(G)$ is a **twig** if either $|\delta(X)| = 2$, or $|\delta(X)| = 3$ and $|X| \geq 5$.

**Lemma:** Every twig is a burl.

**Proof:** $\Pr[|M_w \cap \delta(X)| = 1] = 1$. 

![Diagram](image-url)
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One of the edges of $\delta(X)$ is in at least 2 perfect matchings of the new graph.
Lemma: Vertex set of any cycle of length 4 is a burl.

Proof: \[ \mathbb{E}[\mid M_w \cap \delta(X) \mid] = \frac{4}{3}, \]
\[ \mid M_w \cap \delta(X) \mid \in \{0, 2, 4\}, \]
\[ \Pr[\mid M_w \cap \delta(X) \mid = 0] \geq \frac{1}{3}. \]
We say that $G_1$ and $G_2$ are obtained from $G$ by a cut contraction. (We can apply a similar procedure to cuts of size 2.)

**Lemma:** $m^*(G) \geq m^*(G_1)m^*(G_2)$, $f(G) \geq f(G_1) + f(G_2) - 2$.

We say that $G$ has a core if a graph $G'$ with $|V(G')| \geq 6$ and no non-trivial cuts of size at most 3 can be obtained from $G$ by a (possibly empty) sequence of cut contractions.
Main technical statement

**Theorem**: Let $G$ be a cubic bridgeless graph then, if $G$ has a core

$$m^*(G) \geq 2^\alpha |V(G)| - \beta f(G) + \gamma,$$

where $\alpha << \beta << \gamma << 1$. 
**Main technical statement**

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$$m^*(G) \geq 2^{\alpha |V(G)| - \beta f(G) + \gamma}.$$ 

**Sketch of a proof:** By induction.

1. If $G$ has no non-trivial cuts of size at most 3 apply Voorhoeve’s splitting trick.

   $$3m^*(G) \geq m^*(G_1) + m^*(G_2) + m^*(G_3) + m^*(G_4)$$

   $$|V(G_i)| = |V(G)| - 2$$

   $$f(G_i) \leq f(G) + 2$$
Main technical statement

**Theorem**: Let $G$ be a cubic bridgeless graph then, if $G$ has a core

$$m^*(G) \geq 2^{\alpha|V(G)|-\beta f(G)+\gamma}.$$ 

**Sketch of a proof**: By induction.

1. If $G$ has no non-trivial cuts of size at most 3 apply Voorhoeve’s splitting trick.

2. Easy if for some small cut both contractions $G_1$ and $G_2$ have a core

$$m^*(G) \geq m^*(G_1)m^*(G_2) \geq 2^{\alpha|V(G)|-\beta f(G)-2\beta+2\gamma}$$

$$f(G) \geq f(G_1) + f(G_2) - 2.$$
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1. If $G$ has no non-trivial cuts of size at most 3 apply Voorhoeve’s splitting trick.

2. Easy if for some small cut both contractions $G_1$ and $G_2$ have a core.

3. Otherwise, $G$ has a tree structure with respect to small cuts with exactly one “large” piece.
Cut decomposition

$G$ has a tree structure with respect to small cuts with exactly one “large” piece.

$m^*(G) \geq m^*(G') \geq 2^{\alpha |V(G')| - \beta f(G') + \gamma}$
Cut decomposition

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If this part contained no long paths then we lost $k$ twigs by deleting it a multiple of $k$ vertices and gained at most one twig.
Burls in long paths of 3-cuts

A burl
Burls in long paths of 3-cuts
Burls in long paths of 3-cuts

Behaves like a 4-cycle
Conjecture (Lovász, Plummer, 1986): There exist constants $c_1(k), c_2(k) > 0$ such that for every $k$-regular graph $G$ with $m^*(G) \geq 1$, we have

$$m(G) \geq c_1(k) \left( c_2(k) \right)^{|V(G)|}$$

Moreover, $c_2(k) \to 1$ as $k \to 1$.

Counterexample (Geelen, N.): For $k \geq 4$ there exist $k$-regular graphs $G$ with $m^*(G) \geq 1$, and

$$m(G) \leq 2^{O(\sqrt{|V(G)|})}$$

(Examples are not $(k-1)$-edge-connected.)
**$k$-regular graphs**

**Conjecture (Lovász, Plummer, 1986):** There exist constants $c_1(k), c_2(k) > 0$ such that for every $k$-regular $(k-1)$-edge-connected graph $G$ we have

$$m(G) \geq c_1(k) \left( c_2(k) \right)^{|V(G)|}$$

Moreover, $c_2(k) \to 1$ as $k \to 1$.

**Theorem (Seymour):** There exist a constant $\epsilon > 0$ such that $m(G) \geq 2^{\epsilon|V(G)|}$ in every $k$-regular $(k-1)$-edge-connected graph $G$. 
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\[
m(\overline{G}) \geq c_1(k) \left( c_2(k) \right)^{|V(\overline{G})|}
\]

Moreover, \( c_2(k) \to 1 \) as \( k \to 1 \).

Theorem (Seymour): There exist a constant \( \varepsilon > 0 \) such that \( m(\overline{G}) \geq 2^{\varepsilon |V(\overline{G})|} \) in every \( k \)-regular \((k-1)\)-edge-connected graph \( G \).

Proof: Consider \( w \equiv 1/k \in PMP(G) \).
Choose 3-perfect matchings independently from the corresponding distribution.
Thank you!