Goal: understand $\mathrm{End}_{\mathfrak{g}} P(\mu)$ for $\mu \in -P_+ \setminus \{0\}$
"biggest proj."

Christmas gift = one of Soergel's big results
but "without" Soergel bimodules!

maybe singular
Fix $\lambda \in \mathfrak{h}^*_{dom}$ integral and dominant. Let B_λ be finite-dim'l algebra with $\mathcal{O}_\lambda \simeq B_\lambda\text{-mod}$.
simples indexed by W

Thm. (Struktursatz) Projective P_{w_0} is unique indecomposable projective-injective in B_λ-mod.
Functor $\mathbb{V} = \mathrm{Hom}_{B_\lambda}(P_{w_0}, -)$ fully faithful on B_λ-proj.

This result actually consequence of stronger one.
Let $C = \mathrm{End}_{B_\lambda}(P_{w_0})$.

Thm. (Double-centralizer) As algebras,

$$A := \mathrm{End}_C(P_{w_0}) \simeq B_\lambda$$

Rem. Suppose double centralizer and fix $e \in B_\lambda$ idempotent with $P_{w_0} \simeq B_\lambda e$. Then, $C \simeq eB_\lambda e$ and have equivalence

$$\begin{array}{c} \mathrm{add}(B_\lambda e) \xrightarrow[\sim]{\mathrm{Hom}_{eB_\lambda e}(B_\lambda e, -)} \mathrm{proj}\, A \xrightarrow[\sim]{\text{Thm}} \mathrm{proj}\, B_\lambda \\ \cap \\ eB_\lambda e\text{-mod} \end{array}$$

[AC, Cor. V.1.4]

of quasi-inverse $eB_\lambda \otimes_{B_\lambda} - \simeq \mathrm{Hom}_{B_\lambda}(P_{w_0}, -) = \mathbb{V}$.
[AC, Prop. V.1.7]

(ex: $\mathfrak{g}=\mathfrak{sl}_2$) Take $\lambda=0$. Then $B_0 = \begin{tikzcd} \underset{0}{\bullet} \arrow[r, bend left, "\alpha"] & \underset{-2}{\bullet} \arrow[l, bend left, "\beta"] \end{tikzcd} / \langle \beta\alpha = 0 \rangle$

$\Delta(0) = \begin{smallmatrix} 0 \\ -2 \end{smallmatrix} \simeq P(0)$ $\quad B_0\varepsilon_{-2} = P(-2) = \begin{smallmatrix} -2 \\ 0 \\ -2 \end{smallmatrix} \simeq P(-2)^* \simeq I(-2)$

graded dual + chevalley

$\operatorname{End}_{\mathfrak{g}} P(-2) = \operatorname{sp}_{\mathbb{C}}\{ \mathrm{id},\ x : P(-2) \twoheadrightarrow L(-2) \hookrightarrow \Delta(0) \hookrightarrow P(-2) \}$

(dim=2, comm.) $\simeq \mathbb{C}[x]/x^2\mathbb{C}[x] = C$ coinvariant alg of type A_1 (Endomorphismensatz)

Also, $\mathbb{V} = \operatorname{Hom}_{\mathfrak{g}}(P(-2), -) : \mathcal{O}_0 \longrightarrow \operatorname{Mod} C$ verifies

exact! action by precomposition

$\mathbb{V}(P(-2)) = C$

$\mathbb{V}(P(0)) = \operatorname{Hom}_{\mathfrak{g}}(P(-2), P(0)) \simeq \operatorname{Hom}_{\mathfrak{g}}(P(-2), L(-2)) \simeq xC$ (dim=1)

and

$$\operatorname{End}_{\mathfrak{g}} P(0) \xrightarrow{\mathbb{V}} \operatorname{End}_C(xC)$$

trivially bijective; dim=1; dim=1

$$\operatorname{Hom}_{\mathfrak{g}}(P(0), P(-2)) \xrightarrow{\mathbb{V}} \operatorname{Hom}_C(xC, C)$$

dim=1 bijective by exactness; dim=1 as $x \mapsto 1$ does not define morphism

with finally

$$\begin{tikzcd} \operatorname{Hom}_{\mathfrak{g}}(P(-2), P(\lambda)) \arrow[r, "\mathbb{V}"] \arrow[d, phantom, "\parallel"] & \operatorname{Hom}_C(C, \mathbb{V}(P(\lambda))) \arrow[d, phantom, "\wr\!\mid"] \\ \mathbb{V}(P(\lambda)) \arrow[r, "\mathrm{id}"] & \mathbb{V}(P(\lambda)) \end{tikzcd} \quad \text{for } \lambda \in \{0, -2\}$$

bijective!

Gives Struktursatz. Iso. $B_0 \simeq \operatorname{End}_{\varepsilon_{-2}B_0\varepsilon_{-2}}(P(-2))$ given by

$$\begin{pmatrix} \overset{\varepsilon_{-2}}{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \overset{\alpha}{a_{21}} & \overset{\varepsilon_0}{a_{22}} & \cancel{a_{23}} \\ a_{31} & \underset{\beta}{a_{32}} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

if and only if $a_{11} = a_{33}$ and $a_{12} = a_{13} = a_{23} = 0$

§ Injectivity for projectives in $\mathcal{O}$

Fact: $\operatorname{soc}\Delta(\lambda)$ is irreducible with antidominant highest ℓ-weight μ for all $\lambda \in \mathfrak{h}^*$

Humphreys Prop. 4.1, Thm. 4.2 (c) and Thm. 4.8

Fix $\lambda \in \mathfrak{h}^*_{dom}$ integral and dominant. Then, unique antidominant weight in $W \cdot \lambda$ is $w_0 \cdot \lambda$. Thus, for all $\mu \in W \cdot \lambda$,

$$\operatorname{soc}\Delta(\mu) \simeq L(w_0 \cdot \lambda)$$

and

$$M \in \mathcal{F}(\Delta) \cap \mathcal{O}_\lambda \rightsquigarrow \operatorname{soc} M \in \operatorname{add}(L(w_0 \cdot \lambda))$$

Now, recall that

essentially translation (using translation functors) of result in Alexis's talk

$W_{-\rho} = W \Rightarrow$ 1 simple $\operatorname{Ext}^1_{\mathfrak{g}}(L(\lambda), L(\lambda)) = 0$

Prop. $P(w_0 \cdot \lambda)^* \simeq P(w_0 \cdot \lambda)$

Proof. $P(\mu) \simeq \theta_{-\rho,\lambda}(\Delta(-\rho))$ with $\Delta(-\rho) \simeq L(-\rho)$ self-dual and $\theta_{-\rho,\lambda}$ compatible with duality. $\square$

Hence, $P(w_0 \cdot \lambda) \simeq I(w_0 \cdot \lambda)$ is projective-injective. We also have the reciprocal.

Prop. For $\mu \in W \cdot \lambda$, $P(\mu)$ injective $\Rightarrow \mu = w_0 \cdot \lambda$.

Proof. $P(\mu)$ inj. $\Rightarrow P(\mu) =$ inj. env. of $\operatorname{soc} P(\mu)$

$\operatorname{soc}(P(\mu)) \in \operatorname{add}(L(w_0 \cdot \lambda))$ by the above

$\Rightarrow P(\mu) \in \operatorname{add}(I(w_0 \cdot \lambda)) = \operatorname{add}(P(w_0 \cdot \lambda))$

both $P(\mu), P(w_0 \cdot \lambda)$ are indecomposables

$\Rightarrow P(\mu) \simeq P(w_0 \cdot \lambda) \Rightarrow \mu = w_0 \cdot \lambda$. $\square$

take top

Thus, $P(w_0 \cdot \lambda)$ = unique indecomposable projective-injective module in $\mathcal{O}_\lambda$. Also, $P(w_0\cdot\lambda) = T(\lambda)$ tilting so $\mathrm{Coker}(\Delta(\lambda) \hookrightarrow P(w_0\cdot\lambda)) \in \mathcal{F}(\Delta) \cap \mathcal{O}_\lambda$ and injective envelope of this cokernel hence lie in $\mathrm{add}(P(w_0\cdot\lambda))$. Corresponding injective resolution of $\Delta(\lambda)$ looks like

$$0 \to \Delta(\lambda) \to P(w_0\cdot\lambda) \to X$$

i.e. is direct sum of copies of $P(w_0\cdot\lambda)$...

with $X \in \mathrm{add}(P(w_0\cdot\lambda))$. Using the indecomposable functors θ_w ($w \in W$) discussed with Alexis, we get

$$\begin{array}{ccccccc} 0 \longrightarrow & \theta_w \Delta(\lambda) & \longrightarrow & \theta_w P(w_0\cdot\lambda) & \longrightarrow & \theta_w X \\ & \wr\wr & & \| & & \| \\ & P(w\cdot\lambda) & & X_{1,w} & & X_{2,w} \end{array}$$

with $X_{1,w}, X_{2,w}$ both projective-injective.

For B_λ with $\mathcal{O}_\lambda \simeq B_\lambda$-mod, above says that every projective $P \in \mathrm{mod}\, B_\lambda$ lies inside sequence

$$0 \to P \to X_1 \to X_2$$

with X_1, X_2 projective-injective. Hence, take $P = B_\lambda$. $\mathrm{domdim}\, B_\lambda \geq 2$.

Lemma: Only indecomposable $P_{w_0} \in \mathrm{proj}\, B_\lambda \cap \mathrm{inj}\, B_\lambda$ faithful (i.e. $\forall a,b \in B_\lambda, \exists x \in P_{w_0}$ s.t. $ax \neq bx$).

Proof. Fix $a,b \in B_\lambda$ s.t. $ax = bx$ for all $x \in P_{w_0}$ with $n \in \mathbb{N}$ such that $B_\lambda \hookrightarrow P_{w_0}^{\oplus n}$. Take $x_1, \dots, x_n \in P_{w_0}$ so that $1 \mapsto (x_1, \dots, x_n)$ by inclusion. Then $a = b$. $\boxtimes$

Notation Algebra with faithful proj-inj. module is said to be (left) QF-3.

Rem. From now on can take any finite-dim'l alg. B_λ with domdim $B_\lambda \geq 2$ and (unique) proj.-inj. P_{w_0} which is faithful. ($C = \mathrm{End}_{B_\lambda} P_{w_0}$, $A = \mathrm{End}_C P_{w_0}$)

§ Change of paradigm

Let $T :=$ (injective hull of B_λ) $= P_{w_0}^{\oplus n}$ for some n.
Let also

$$\mathcal{G} = \mathrm{End}_{B_\lambda} T \quad \text{and} \quad \mathbb{A} = \mathrm{End}_{\mathcal{G}} T.$$

Have exact sequence

$$0 \to B_\lambda \xrightarrow{\delta} T \xrightarrow{\varepsilon} T^{\oplus N} \quad \text{for some } N \in \mathbb{N}$$

Will show $B_\lambda \simeq \mathbb{A}$. Equivalent to wanted result as:

Prop Let $C = \mathrm{End}_{B_\lambda}(P_{w_0})$. Then, as algs,

$$\underbrace{\mathrm{End}_{\mathcal{G}} T}_{\mathbb{A}} \simeq \underbrace{\mathrm{End}_C P_{w_0}}_{A}$$

$(p_i f q_j)(p_i g q_j) = (x_{ij})$
$x_{ij} = \sum_k p_i f q_k p_k g q_j = p_i f g q_j$

$f \mapsto (p_i f q_j)_{i,j=1}^n$

Pf. $\mathcal{G} = \mathrm{End}_{B_\lambda}(P_{w_0}^{\oplus n}) \simeq M_n(\mathrm{End}_{B_\lambda}(P_{w_0})) = M_n(C)$

$(f_{i,j})_{i,j=1}^n \mapsto \sum_{i,j} q_i f_{i,j} p_j$ (inverse) alg. morph

Thus, $\mathcal{G}$ and C Morita-equivalent.
Also, equivalence $\mathrm{mod}\, C \simeq \mathrm{mod}\, \mathcal{G}$ sends P_{w_0} to T and result follows. ☒

Rem. $B_\lambda \subseteq \mathbb{A}$ as T is faithful

$t \mapsto bt$ verifies
$f \cdot t = f(t) \mapsto bf(t) = f(bt)$
$= f \cdot (bt)$

§ Crux

Let $\mathcal{C}_0 = \{f \in \mathcal{C} \mid f(B_\lambda) = 0\}$ and $\quad {\scriptstyle 0 \to B_\lambda \to T}$

$$Q_{tot} = \bigcap_{f \in \mathcal{C}_0} \ker f \supseteq B_\lambda .$$

NON-TRIVIAL

Lemma $\mathbb{A} \simeq Q_{tot}$ as B_λ-modules.

$\hookrightarrow (b \cdot f)(g \cdot t) = b(f(g \cdot t)) = b(g \cdot f(t)) = bg(f(t)) = g(bf(t))$
$\quad {\scriptstyle \in \mathbb{A} \quad \in \mathcal{C}} \qquad\qquad = g \cdot ((b \cdot f)(t))$

Corollary $B_\lambda \simeq \mathbb{A} \simeq A$

Pf. Recall exact sequence $0 \to B_\lambda \xrightarrow{\delta} T \xrightarrow{\varepsilon} T^{\oplus N}$ and let $p_i : T^{\oplus N} \to T$ be ith-projection $(1 \le i \le N)$. Then

$$Q = \bigcap_{i=1}^{N} \ker(p_i \circ \varepsilon) = \ker \varepsilon = B_\lambda$$

$\llcorner q \in Q \rightsquigarrow 0 = \sum_i (\iota_i \circ p_i \circ \varepsilon)(q)$
w/ $\iota_i : T \to T^{\oplus n}$ ith inclusion

so that

$$B_\lambda \subseteq \mathbb{A} \simeq Q_{tot} \subseteq Q = B_\lambda \qquad \boxtimes$$

We now prove non-trivial lemma.

Pf. Map $\mathcal{B} : \mathcal{C} \to T$ given by $\mathcal{B}(f) = f(1)$ is surj. as

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{Hom}_{B_\lambda}(\delta, T)"] \arrow[dr, "\mathcal{B}"'] & \mathrm{Hom}_{B_\lambda}(B_\lambda, T) \arrow[d, "\mathrm{ev}_1", "\wr"'] \\ & T \end{tikzcd}$$

commutes + $\mathrm{Hom}_{B_\lambda}(\delta, T)$ surj. since T injective.
(and δ injective)

Now, $\gamma : \mathbb{A} = \mathrm{End}({}_{\mathcal{G}}T) \longrightarrow T$ given by $\gamma(f)=f(1)$ is injective. Indeed, fix $f \in \mathbb{A}\setminus\{0\}$. Then, $f(t)\neq 0$ for some $t\in T$. By above, $t=\mathcal{B}(g)=g(1)$ with $g\in\mathcal{G}$ and

$$0\neq f(t)=f(g(1))=f(\overset{\in\mathcal{G}}{g}\cdot 1)=g\cdot f(1)=g\cdot\gamma(f)$$

gives $\gamma(f)\neq 0$ as claimed. Also, γ clearly B_λ-lin.

$\gamma(bf)=(bf)(1)=bf(1)=b\gamma(f)$

Want to show $A\simeq \mathrm{Im}\,\gamma = Q_{tot}=\bigcap_{g\in\mathcal{G}_0}\ker f\subseteq T$.

($\subseteq$) Fix $g\in\mathcal{G}_0$ and $f\in A$. Then

$$g(\gamma(f))=g(f(1))=g\cdot f(1)\underset{f\ \mathcal{G}\text{-lin.}}{=}f(g\cdot 1)=f(g(1))\underset{g\in\mathcal{G}_0}{=}0.$$

($\supseteq$) Fix $q\in Q_{tot}$. Define $\psi:\mathcal{G}\longrightarrow T$ by $\psi(f)=f(q)$. Then $\psi(\mathcal{G}_0)=0$. Also, for $f\in\mathcal{G}$, $\mathcal{B}(f)=f(1)=0$ iff $f(B_\lambda)=0$, that is iff $f\in\mathcal{G}_0$. Thus

$$\ker\mathcal{B}=\mathcal{G}_0\subseteq\ker\psi$$

$\mathcal{B}(g\cdot f)=\mathcal{B}(g\circ f)=g(f(1))=g\cdot\mathcal{B}(f)$

with $\mathcal{B},\gamma$ easily seen to be $\mathcal{G}$-linear. Passing to cokernels, $\exists!\,\mu: T\to\mathrm{Im}\,\psi\subseteq T$ $\mathcal{G}$-linear s.t. $\mu\circ\mathcal{B}=\psi$. Hence, $\mu\in\mathbb{A}$ and

$$\gamma(\mu)=\mu(1)=\mu(\mathcal{B}(id))=\psi(id)=q \qquad \boxtimes$$

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{G} \arrow[r, "\mathcal{B}"] \arrow[d, equal] & T \arrow[r] \arrow[d, dashed, "\exists!\mu"] & 0 \\
0 \arrow[r] & \ker\psi \arrow[r] & \mathcal{G} \arrow[r, "\psi"] & \mathrm{Im}\,\psi \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$