Monoidal categorifications of cluster algebras

What? Where? Why? How?

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later, possibly with coefficients...

Fix A = cluster algebra over $\mathbb{Z}$

subalgebra of $\mathbb{Z}(x_1, \dots, x_n)$ for some n, depending on quiver Q and whose generators = cluster variables are grouped in clusters of size n and are constructed recursively, via mutation, from initial seed $(Q, (x_1, \dots, x_n))$

Important notion in combinatorics, Poisson geometry, discrete dynamical systems, rep. theory, commutative/non-commutative algebraic geometry, ...

Fact: For a cluster algebra (associated to a quiver) an interesting linearly independent set is given by cluster monomials

product of non-negative powers of cluster variables of a given/fixed cluster

Constructing 'canonical bases' for cluster algebras (that contain the cluster variables) is an important & largely open problem.

see Keller's 2014 "Cluster algs & cluster mon".

ex: reps of finite gp, of finite-dim'l Lie alg., of q-gps...

Fix now $\mathscr{C}$ an abelian monoidal category. Then, Grothendieck group

$K_0(\mathscr{C})$ = abelian group generated by equivalence classes $\{[V] \mid V \in \mathscr{C}\}$
via relation $[V]=[S]+[Q]$ if exists SEC $0 \to L \to V \to Q \to 0$ in $\mathscr{C}$

is a ring:

$$[V][W] = [V \otimes W]$$

and has a canonical $\mathbb{Z}$-basis given by the classes of simple objects.

corresponding structure constants are positive
(= multiplicities of simple composition factors in $\otimes$)
= Clebsch-Gordan coefficients

Then, our seminar aims to study the following:

Def. (Hernandez–Leclerc, ≈2009) For A and $\mathscr{C}$ as above, we say that $\mathscr{C}$ is a monoidal categorification of A if there's a ring. iso.

MC hereafter

$$\varphi : A \longrightarrow K_0(\mathscr{C})$$

What?

that sends cluster monomials to classes of simple objects.

Rem. Suppose that $\mathcal{C}$ is a MC of A. Then, the canonical basis of $K_0(\mathcal{C})$ (given by the classes of simple modules) give a canonical basis for A (with positive structure constants). Thus, a MC $\mathcal{C}$ of A gives non-trivial info. on A. Also give non-trivial info on $\mathcal{C}$ as then

Why?

mutation/exchange relations in A $\overset{\varphi}{\longleftrightarrow}$ relations in $K_0(\mathcal{C})$ $\longleftrightarrow$ exact sequences in $\mathcal{C}$

Now, fix $\mathcal{C}$, A and φ as in the definition and let $x \in A$ be a cluster monomial. Then, $\varphi(x) = [V]$ for some simple object $V \in \mathcal{C}$. Also, x^2 is also a cluster monomial so $\varphi(x^2) = [V]^2 = [V \otimes V]$ is the class of a simple object of $\mathcal{C}$. It follows that

φ sends cluster monomials to real simple objects
└ i.e. of tensor square simple

Moreover, in the initial def of HL, a MC $\varphi: A \to K_0(\mathcal{C})$ was meant to induce bijection between cluster monomials and classes of real obj. This is a strong condition.

Now, few abelian monoidal categories have non-trivial real simples obj. For example, let $\mathcal{C}$ be the category of finite-dim'l reps of a group G or of a finite-dim'l Lie algebra $\mathfrak{g}$. Then, for two objects $V, W \in \mathcal{C}$,

$$G \curvearrowright V \otimes W \text{ via } g\cdot(v\otimes w) = gv \otimes gw$$
$$\mathfrak{g} \curvearrowright V \otimes W \text{ via } x\cdot(v\otimes w) = gv \otimes w + v \otimes gw$$

and, in both cases, we have isomorphism $\tau_{V,W} : V\otimes W \to W \otimes V$ given by

$$\tau_{V,W}(v\otimes w) = w \otimes v.$$

In particular, for any simple $V \in \mathcal{C}$, $\tau_{V,V}$ is an automorphism of $V\otimes V$ which is'nt a multiple of the identity (if $\dim V \neq 1$). Hence,

Schur's lemma $\longrightarrow$ all simple real objects of $\mathcal{C}$ are 1-dim'l

The same reasoning can be applied to finite-type quantum groups (and any braided (abelian) monoidal category for which the braiding $R_{V,V} : V\otimes V \xrightarrow{\sim} V \otimes V$ is typically non-trivial)

So where can we find MCs "in nature"?

Where: representations of quantum affine algebras / Yangians — Hernandez-Leclerc
of (quiver)-Hecke algebras — Leclerc
Kang-Kashiwara-Kim-Oh-Park
or in geometry
→ Nakajima (perverse sheaves on graded quiver var.)
→ Cautis-Williams (perverse coherent sheaves on affine grassmanians).

In any case, if $\mathcal{C}$ is a MC of A, then there should be objects $V, W \in \mathcal{C}$ with $V \otimes W \not\cong W \otimes V$ (even if $K_0(\mathcal{C}) \simeq A$ is commutative).
MC is sign of $\mathcal{C}$ not braided, but "almost" (see Hernandez's recent review)

ex: $\mathcal{U}_q\hat{\mathfrak{sl}}_2$ = simplest q-affine algebra
Have 1-parameter family of surjective algebra morphisms

$$ev_a : \mathcal{U}_q\hat{\mathfrak{sl}}_2 \twoheadrightarrow \mathcal{U}_q\mathfrak{sl}_2 \qquad (a \in \mathbb{C}^\times)$$

$V(n,a) = ev_a^*$ (unique $(n+1)$-dim'l irrep of $\mathcal{U}_q\mathfrak{sl}_2$) $\quad (n>0)$.
Fix $\mathcal{C}$ = monoidal Serre subcat. of $\mathcal{U}_q\hat{\mathfrak{sl}}_2$-mod generated by $V(1,1)$, $V(1,q^2)$
closed under subobj./quotients/extensions

Then, we have exact sequences

$$0 \to \mathbb{1} \to V(1,q^2) \otimes V(1,1) \to V(2,1) \to 0$$

trivial rep

[$\mathbb{1}$]

$$0 \to V(2,1) \to V(1,1) \otimes V(1,q^2) \to \mathbb{1} \to 0$$

so $[V(1,1)][V(1,q^2)] = 1 + [V(2,1)] = [V(1,q^2)][V(1,1)]$ in $K_0(\mathscr{C})$. This is the (only) exchange relation of an A_1-cluster algebra

$$V(1,1) \longrightarrow V(2,1) \quad \xleftrightarrow[V(1,1)]{\text{mutation at}} \quad V(1,q^2) \longleftarrow V(2,1)$$

and $\mathscr{C}$ actually is a monoidal categorification of this cluster alg. This can be used to show that the (infinitely-many) simple objects of $\mathscr{C}$ all are tensor products of the 3 irreps $V(1,1), V(1,q^2), V(2,1)$!

Rem. Both above SEC come from "renormalized R-matrices"

$$\begin{tikzcd}
0 \arrow[r] & V(2,1) \arrow[r] & V(1,1) \otimes V(1,q^2) \arrow[r] \arrow[dr] & V(1,q^2) \otimes V(1,1) \arrow[d] & \\
& & & \mathbb{1} \arrow[dr] & \\
& & & & 0
\end{tikzcd}$$

and similarly for other one...

Because proving that a category $\mathscr{C}$ is a MC of a cluster alg. A is a priori super hard.

Indeed, to show that $\varphi: A \longrightarrow K_0(\mathscr{C})$ is a ring iso., one may think that all mutation relations of A must be verified in $K_0(\mathscr{C})$. However, it is already hard (if even possible) to guess what these mutation relations look like in A (as they involve cluster variables constructed iteratively) so the above task seems totally unreasonable. Fortunately,

read "I am not sure about this" How?

Thm. (KKKOP, "naive version") Suppose $\mathscr{C}$ admits renormalized R-matrices. Then, under some conditions, it suffices to look at 1-step mutations to deduce that $\mathscr{C}$ is a MC of some cluster alg. A.

The above theorem played a key role in the work of Cautis-Williams on the affine grassmannians. I would like to understand it better. Another thing I would like to better understand is the relationship between additive & monoidal categorification (see Fujita's work, etc.).

↳ for another time...

To conclude this exposition, let us mention that, in the original definition of MC, it is asked that the isomorphism $\varphi: A \to K_0(\mathscr{C})$ sends cluster variables to classes of prime real simple objects.

no non-trivial tensor factorisation

This can be seen as an analog of "maximality" for MCs and can lead in practice to complete characterization of prime real simple objects in $\mathscr{C}$ (see for ex. Chari's recent work).

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Higher order KR-modules, ...