Introduction to cluster algebras

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Recall from first meeting

$\mathbb{A}$ = cluster alg. of rank $n \approx$ subalg. of $\mathbb{Z}(x_1, \ldots, x_n)$ depending on
Q = quiver whose generators = cluster variables are grouped in clusters of size n and are constructed recursively, via mutation, from initial seed $(Q, (x_1, \ldots, x_n))$
$\hookrightarrow$ have interesting LI elements = cluster monomials
$= \bigcup_{\text{cluster } C} \{\text{monomials in the cluster variables of } C\}$

$K_0(\mathcal{C})$ = Grothendieck ring of abelian monoidal category $\mathcal{C}$
(ex: finite-dim'l repns of $Y(\mathfrak{sl}_2) = Y_0(\mathfrak{sl}_2) \ldots$)
$\hookrightarrow$ have "canonical basis" consisting of equivalence classes of simple objects

Then,

"$\mathcal{C}$ = monoidal categorification of $\mathbb{A}$"
$\iff$ "there's ring iso. $\varphi: \mathbb{A} \to K_0(\mathcal{C})$ sending cluster monomials to equivalence classes of simples"
$\iff$ "there's ring iso. $\varphi: \mathbb{A} \to K_0(\mathcal{C})$ compatible with our two remarkable sets of LI elements"

Why?
- informations on $\mathbb{A}$ ("canonical basis associated to positive structure constants, ...)
- informations on $\mathcal{C}$ (" relations in K_0, exact sequences, prime/real simple objects, ...)

Today: introduction to (part of) the cluster algebra part of the story...

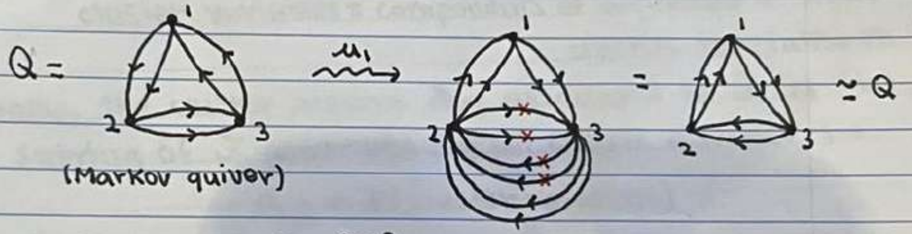
Q = (isoclass of) (finite) quiver with no 1-cycles or 2-cycles
$= (\underbrace{Q_0}_{\text{nodes}}, \underbrace{Q_1}_{\text{arrows}}, s, t)$ $\qquad s: Q_1 \to Q_0$ "source map" $\qquad t: Q_1 \to Q_0$ "target map"

For $k \in Q_0$, define quiver $\mu_k(Q)$ from Q following
(1) Add $i \to j$ for all $i \to j \to k$ in Q "add shortcuts"
(2) Reverse all $i \to k$ and $k \to j$ in Q "swap incident arrows"
(3) Erase all 2-cycles created from (1) & (2) "save yourself"

__Rem.__ $\mu_k(\mu_k(Q)) = Q$

__Def.__ Q, Q' mutation equivalent $\Leftrightarrow$ linked by finite sequence of mutations

__ex.__



(Markov quiver)

so mutation class of $Q = \{Q\}$.

Now, ~~[illegible]~~ fix $n \in \mathbb{N} \setminus \{0\}$ and $\mathcal{F} = \mathbb{Z}(x_1, \dots, x_n)$. Then a seed is a pair (Q, u) with
Q = quiver such that $|Q_0| = n$
$u = (u_1, \dots, u_n) \in \mathcal{F}^n$ free with $\mathcal{F} \simeq \mathbb{Z}(x_1, \dots, x_n)$

We can mutate seeds : fix $k \in \{1, \dots, n\}$

$$(Q, u) \longmapsto \mu_k(Q, u) = (\underbrace{\mu_k(Q)}_{\text{as before}}, u')$$

__Rem.__ $\mu_k(\mu_k(Q,u)) = (Q,u)$

with $u' = (u_1, \dots, u_{k-1}, u'_k, u_{k+1}, \dots, u_n)$ and

(exchange relation) $$u_k u'_k = \prod_{\substack{\alpha \in Q_1 \\ s(\alpha) = k \\ \text{(outgoing)}}} u_{t(\alpha)} + \prod_{\substack{\alpha \in Q_1 \\ t(\alpha) = k \\ \text{(incoming)}}} u_{s(\alpha)}$$

ex: $Q = \overset{1}{\bullet} \to \overset{2}{\bullet}$ $\quad u = (x_1, x_2) \xrightarrow[\text{notation}]{\text{compact}} \overset{x_1}{\bullet} \to \overset{x_2}{\bullet} = (Q, u)$

We have

$$\begin{tikzcd}
\frac{1+x_1}{x_2} \to x_1 \arrow[r, squiggly, "\mu_1", "\text{flip}"'] & x_1 \to x_2 \arrow[r, squiggly, "\mu_1"] & \frac{1+x_2}{x_1} \leftarrow x_2 \arrow[r, squiggly, "\mu_1"] \arrow[d, squiggly, "\mu_2"] & x_1 \to x_2 \\
\frac{1+x_1}{x_2} \leftarrow \frac{1+x_1+x_2}{x_1x_2} \arrow[u, squiggly, "\mu_2"] & & \frac{1+x_2}{x_1} \to \frac{1+x_1+x_2}{x_1x_2} \arrow[ll, squiggly, "\mu_1"'] &
\end{tikzcd}$$

$\left(x_1 = \dfrac{x_2+1}{\frac{1+x_2}{x_1}} \right)$

Def. Fix Q quiver with $|Q_0| = n$ and $u = (x_1, \dots, x_n) \in \mathcal{F}^n$.
Then (Q, u) = initial seed.
A cluster associated to Q is a seed (Q', u') mutation-equiv to (Q, u)

cluster variables = components of sequences u' appearing in clusters associated to Q

Finally, the cluster algebra $\mathcal{A}_Q$ associated to Q is the subring of $\mathcal{F}$ generated by all cluster variables, i.e.

$$\mathcal{A}_Q = \mathbb{Z}[\text{cluster variables}]$$

ex: In the example above, the clusters are $\overset{x_1}{\bullet} \to \overset{x_2}{\bullet}$, $\overset{\frac{1+x_2}{x_1}}{\bullet} \text{—} \overset{x_2}{\bullet}$, ...
and $\mathcal{A}_Q = \mathbb{Z}\left[x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\right]$

Rem. ~~[illegible]~~ Let (Q', u') = cluster associated to Q. Then,
$\mathcal{F} \simeq \mathbb{Z}(u_1', \dots, u_n')$ and $\mathcal{A}_Q \simeq \mathcal{A}_{Q'}$ (for $u' = (u_1', \dots, u_n') \in \mathcal{F}^n$).
Also, it is super rare to find a cluster algebra with finitely many cluster variables (=cluster-finite cluster algebra). In the generality in which we're working now this'll only happen when the underlying (undirected) graph of Q is an ADE Dynkin quiver. Then, the "non-trivial" denominators of cluster variables in terms of the variables $x_1, \dots, x_n$ of the initial seed will be in bijection with the positive roots of the underlying root system (α_1, α_2 and $\alpha_1 + \alpha_2$ in type A_2 ...)

Thm. (Laurent phenomenon) (Fomin-Zelevinsky)
Fix (Q', u') a cluster of $\mathbb{A}_Q$ with $u' = (u_1', \dots, u_n') \in \mathcal{F}^n$. Then,

$$\{\text{cluster variables of } \mathbb{A}_Q\} \subseteq \mathbb{Z}[(u_1')^{\pm 1}, \dots, (u_n')^{\pm 1}]$$

Rem. This is non-trivial from the point of view of the exchange relt

$$x_k x_k' = \prod_{\substack{\alpha \in Q_1 \\ s(\alpha) = k}} x_{t(\alpha)} + \prod_{\substack{\alpha \in Q_1 \\ t(\alpha) = k}} x_{s(\alpha)}$$

ex: $x_1 \rightrightarrows x_2 \overset{\mu_1}{\rightsquigarrow} \frac{1+x_2^2}{x_1} \leftleftarrows x_2 \overset{\mu_2}{\rightsquigarrow} \frac{1+x_2^2}{x_1} \rightrightarrows \frac{\left(\frac{1+x_2^2}{x_1}\right)^2 + 1}{x_2} = \frac{1+x_1^2+2x_2^2+x_2^4}{x_1^2 x_2}$

(Kronecker quiver)

$\overset{\mu_1}{\rightsquigarrow}$

... $\overset{\mu_2}{\rightsquigarrow}$ $\frac{3x_2^4+1+2x_1^2+x_1^4+3x_2^2+2x_1^2x_2^2+x_2^6}{x_1^3 x_2^2} = \left(\left(\frac{1+x_1^2+2x_2^2+x_2^4}{x_1^2 x_2}\right)^2+1\right)\frac{x_1}{1+x_2^2} \leftleftarrows \frac{1+x_1^2+2x_2^2+x_2^4}{x_1^2 x_2}$

still a Laurent polynomial in x_1, x_2 with coefficients in $\mathbb{N}$

Fomin-Zelevinsky positivity conj. proven by Lee-Schiffler in 2014 in our degree of generality

Up to flip of Q, the mutation is described by

$$F : \mathcal{F}_*^2 \longrightarrow \mathcal{F}_*^2 \qquad (\mathcal{F}_* = \mathcal{F} \setminus \{0\})$$

$$(a, b) \longmapsto \left(b, \frac{b^2+1}{a}\right)$$

Evaluating $x_1 = x_2 = 1$ amounts to looking at the "orbit" of $(1,1) \in \mathcal{F}^2$ by the maps $F, F^2, F^3, \dots$ This gives

$$(1,1),\ (1,2),\ (2,5),\ (5,13),\ \dots,\ F^k(a,b) = (F_{2k-1}, F_{2k+1}), \dots$$

Fibonacci number

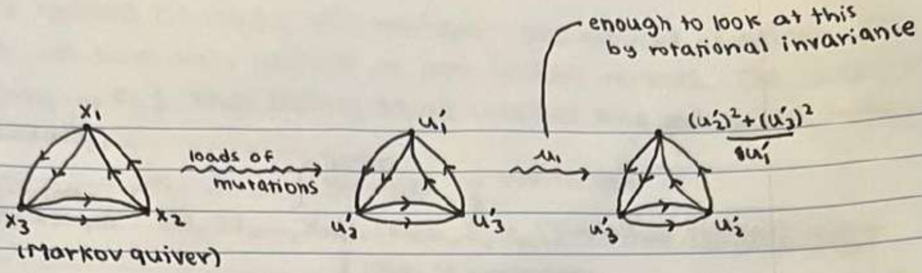
so there are infinitely many cluster vars for $\mathbb{A}_Q$.

Now let us go back to the case of a general quiver Q with $|Q_0| = n$.
Consider

$$\underbrace{\mathcal{U}_Q}_{\text{upper cluster algebra}} = \bigcap_{\substack{(Q', u') \text{ cluster of } \mathbb{A}_Q \\ u' = (u_1', \dots, u_n')}} \mathbb{Z}[(u_1')^{\pm 1}, \dots, (u_n')^{\pm 1}]$$

By the Laurent phenomenon, $\mathbb{A}_Q \subseteq \mathcal{U}_Q$. However this inequality is strict in general... Why care? See later...

ex.



The cluster algebra $\mathbb{A}_Q$ has infinitely many cluster variables here (evaluating at $x_1 = x_2 = x_3 = 1$ give all the solutions of the Markov equation $a^2+b^2+c^2 = 3abc$...). However, giving $\deg x_1 = \deg x_2 = \deg x_3$ and noting that the exchange relt here is homogeneous, one deduces that

$$\deg v = 1 \text{ for all cluster variable } v \in \mathbb{A}_Q$$

Hence $\mathbb{A}_Q$ is $\mathbb{N}$-graded and $w = \frac{x_1^2+x_2^2+x_3^2}{x_1x_2x_3} \notin \mathbb{A}_Q$ (as $\deg = -1$). Nevertheless,

Claim. $w \in \mathbb{U}_Q$.

Pf. One can show by induction that $w = \frac{(u_1')^2+(u_2')^2+(u_3')^2}{u_1'u_2'u_3'}$ for all clusters (Q', u') of $\mathbb{A}_Q$ as

$$\frac{\left(\left(\frac{(u_2')^2+(u_3')^2}{u_1'}\right)^2 + (u_2')^2+(u_3')^2\right)}{\left(\frac{(u_2')^2+(u_3')^2}{u_1'}\right)u_2'u_3'} = \frac{(u_1')^2+(u_2')^2+(u_3')^2}{u_1'u_2'u_3'}$$

\+ rotational invariance... ☐

Fix $1 \leq m \leq n$. Then ice-quiver of type (m,n) is a quiver Q with $|Q_0| = n$ and no $i \to j$ with both $i,j > m$. We call

principal part = full subquiver with vertices $\{1, \dots, m\}$

frozen vertices $= \{m+1, \dots, n\}$

and don't add arrows between frozen vertices

The notions of seed & seed mutation are defined as before, but we can now only mutate at non-frozen vertices. The variables $\{x_{m+1},\dots,x_n\}$ thus belong to all clusters and are called coefficients.

Visually,

$$\underbrace{Q}_{\text{ice-quiver of type }(m,n)},\ \underbrace{(x_1,\dots,x_m)}_{\text{initial variables}},\ \underbrace{(x_{m+1},\dots,x_n)}_{\text{coefficients}}$$

$$(Q, (x_1,\dots,x_m), (x_{m+1},\dots,x_n)) \xrightarrow{\text{loads of mutations}} (Q, (u_1',\dots,u_m'), (x_{m+1},\dots,x_n))$$

$\mathbb{A}_Q = (\mathbb{Z}[\text{coefficients}])[\text{cluster variables}]\dots$

<u>Theorem</u> (Fomin-Zelevinsky). Let

X = rational quasi-affine irreducible variety$/\mathbb{C}$ with $\dim = n$

Q = ice-quiver of type (m,n)

Suppose given functions

$\{\varphi_v \mid v \text{ cluster variable of } \mathbb{A}_Q\}$
$\{\varphi_{x_i} \mid m+1 \le i \le n\}$ $\subseteq \mathbb{C}[X]$ that generate the ring $\mathbb{C}[X]$

Suppose also that $v \mapsto \varphi_v$ and $x_i \mapsto \varphi_{x_i}$ takes exchange relations to equalities in $\mathbb{C}[X]$. Then, this map extends to algebra iso.

$$\varphi: \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{A}_Q \xrightarrow{\sim} \mathbb{C}[X]$$

and we say that $\mathbb{C}[X]$ carries a cluster structure of type Q with initial seed $\{\varphi_{x_i} \mid 1 \le i \le n\}$.

"One might think of a variety with various sets of local coordinates so that each regular function in $\mathbb{C}[X]$ can be expressed locally in terms of the local coordinates. Exchange relations/mutations are then understood as regular changes of local coordinates".

$\hat{X}$ affine cone over the Grassmannian of planes...

We finish by a classical example. Fix $m \ge 1$ and let

$X = Gr_{2,m+3}(\mathbb{C}) = \{V \subseteq \mathbb{C}^{m+3} \mid \dim V = 2\}$ (projective variety, but ok)

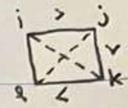
Then, $\mathbb{C}[\hat{X}]$ generated by Plücker coordinates x_{ij} $(1 \le i < j \le m+3)$

homogenous coordinate algebra

subject to $x_{ik}x_{j\ell} = x_{ij}x_{k\ell} + x_{jk}x_{i\ell}$ $(1 \le i < j < k < \ell \le m+3)$.

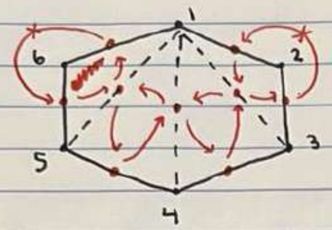
This looks like an exchange relation!

Another way to view the Plücker relation:



"product of diagonals = product of horizontals + product of verticals"

Fix P a polygon with $m+3$ sides with fixed triangulation.
Then $P \rightsquigarrow Q =$ ice quiver of type $(m, 2m+3)$

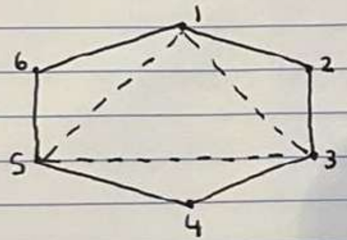


$\rightsquigarrow$

$$\begin{tikzcd}
 & \boxed{16} & & \boxed{12} \arrow[d] & \\
\boxed{56} \arrow[r] & 15 \arrow[u] \arrow[d] & & 13 \arrow[r] \arrow[dl] & \boxed{23} \\
 & \boxed{45} \arrow[r] & 14 \arrow[ul] \arrow[r] & \boxed{34} \arrow[u] &
\end{tikzcd}$$

frozen vertex

mutation can be seen as a "diagonal swap" (Ptolemy's relation)



$\rightsquigarrow$

$$\begin{tikzcd}
 & \boxed{16} & & \boxed{12} \arrow[d] & \\
\boxed{56} \arrow[r] & 15 \arrow[u] \arrow[dr] & & 13 \arrow[r] \arrow[ll] & \boxed{23} \\
 & \boxed{45} & 35 \arrow[l] \arrow[ur] & \boxed{34} \arrow[l] &
\end{tikzcd}$$

<u>Theorem</u> (Fomin-Zelevinsky) $\mathbb{C}[\hat{X}_n]$ carries a cluster algebra of type Q such that

$\{\text{coefficients}\} \overset{1:1}{\longleftrightarrow} \{\text{sides of } P\} \overset{1:1}{\longleftrightarrow} \{\text{corresponding Plückers}\}$

$\{\text{cluster variables}\} \overset{1:1}{\longleftrightarrow} \{\text{diagonals of } P\} \overset{1:1}{\longleftrightarrow} \{\text{'' ''}\}$

$\{\text{clusters}\} \overset{1:1}{\longleftrightarrow} \{\text{triangulations of } P\}$

$\{\text{exchange relations}\} \overset{1:1}{\longleftrightarrow} \{\text{Plücker relations}\}$

· ~~[illegible]~~ We will find a monoidal categorification of this cluster algebra at some point using representations of shifted Yangians associated to $\mathfrak{g} = \mathfrak{sl}_3$. This will give rise to an explicit isomorphism of rings

~~[illegible]~~

$$\mathbb{C}[\hat{X}_n] \simeq K_0(\mathcal{C}_n) \otimes_{\mathbb{Z}} \mathbb{C}$$

for some category $\mathcal{C}_n$ of representations over these shifted Yangians