Outline/Goals:

1) A new (slightly broader) look at Cluster Algebras

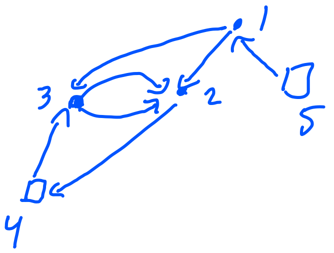
2) c-vectors

3) F-polynomials

Broader look at cluster alg.

We saw cluster algebras as:

Seed $(x_1, \dots, x_n, x_{n+1}, \dots x_{n+m})$ quiver Q (no loops, no 2-cycles)



(By convention, put frozens at end)

and a mutation rule for each:

$x_k x_k' = \prod_{\text{arrows } i \to k} x_i + \prod_{k \to j} x_j$, Q': ① Flip arrows to/from k

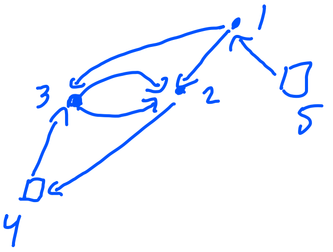
② Add paths: $i \to k$, $k \to j$ in Q $\Rightarrow$ $i \to j$ in Q'

③ "Save yourself": kill 2cycles.

Another way to represent a quiver w/out loops and 2-cycles is as a skew-symmetric matrix B: $B_{ij} := \#$ arrows $i \to j$.

It turns out to be useful to take an $(n+m) \times n$ matrix w/ rows correspond to all variables in our seed, columns only to mutable ones

E.g



$$\rightsquigarrow \quad B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

These matrices in general look like $B = \left(\frac{B_0}{C} \right)$, $B_0^T = -B_0$. We call B_0 the <u>exchange matrix</u>. B is the <u>extended exchange matrix</u>.

B contains the same information as a quiver. To mutate B:

$B' = \mu_k(B) \rightsquigarrow$ ① multiply row, column k by (-1) ← Flip

② $B_{ij} = B_{ij} + \text{sgn}(B_{ik}) \max(B_{ik} B_{kj}, 0)$

↑ Add paths and save yourself

E.g:

$$\begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix} \overset{\mu_1}{\rightsquigarrow} \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & 2+(-1)4 \\ -2 & (-2)+(1)(4) & 0 \end{pmatrix}$$

In this formulation, mutation rule for cluster vars becomes:

$$x_k x_k' = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{jk}<0} x_j^{-b_{jk}}$$

Rmk: I said this is a "generalization". Indeed, everything works if B_0 is skew symmetrizable instead of skew symmetric, but these cluster algebras don't have quivers (they have valued quivers) (But we won't deal w/ these)

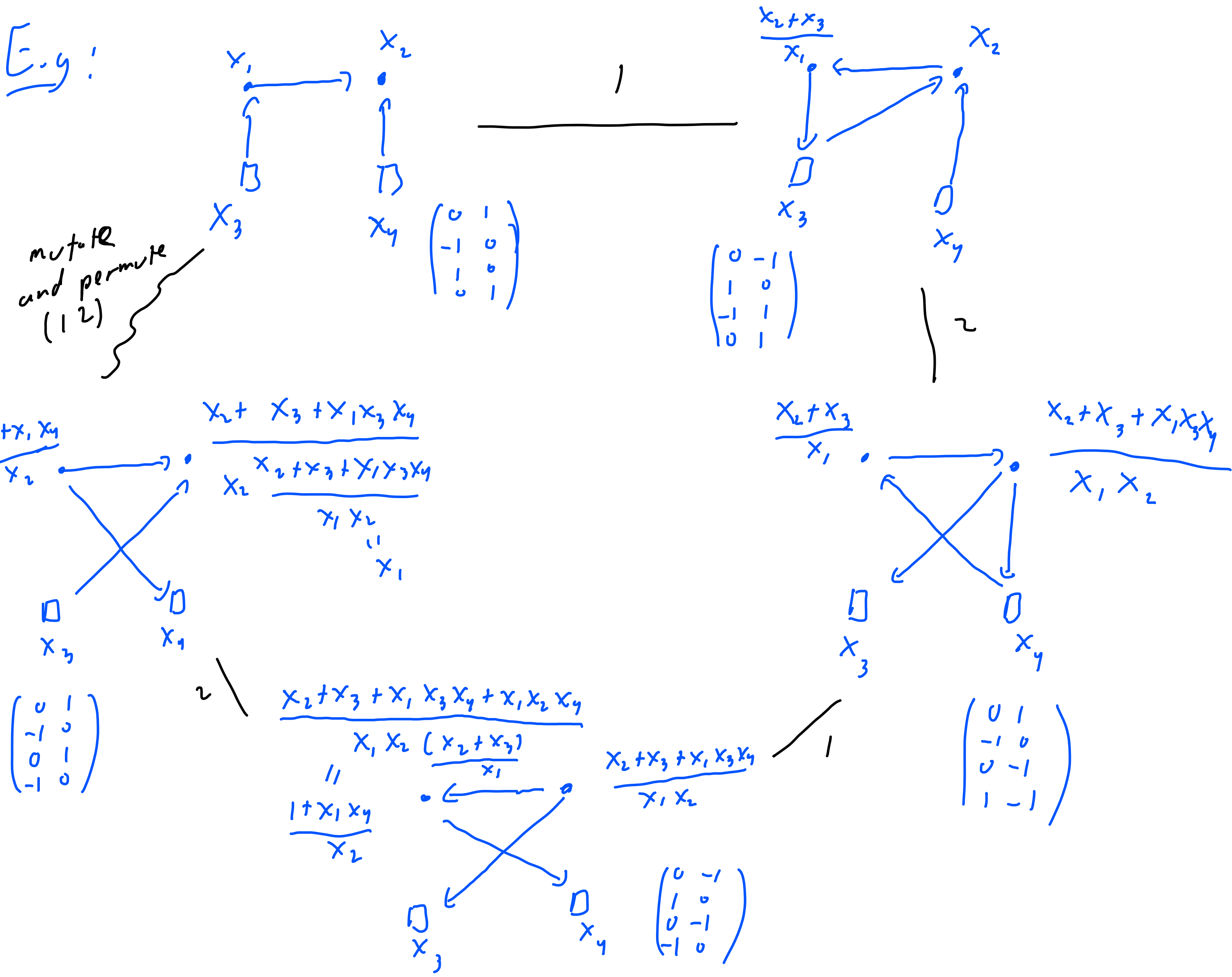
→ There is D diagonal w/ positive entries s.t. DB_0 is skew-symm.

Def: The Exchange Graph of a cluster algebra is the graph with vertices given by equivalence classes of seeds, edges given by mutations. 2 seeds (X, B) and (X', B) are equivalent if X' is a permutation of X and B' is obtained from B by permuting rows and col'ms with that same permutation.

Thm [Gekhtman, Shapiro, Vainshtein]: Suppose B_0 is non-degenerate. Then the exchange graph of $\left(X, \begin{pmatrix} B_0 \\ C \end{pmatrix}\right)$ is fully determined by B_0.

Conjectured for any B_0 by Fomin-Zelevinsky. They proved it earlier for
- finite type
- cluster alg. from triangulation of 2-d surface.

E.g.:



A brief idea about why this happens:

We can encode a seed $(X, B = \binom{B_0}{C})$ by (X, Y, B_0) where Y is a length n vector of Laurent monomials $Y_j = \prod_{i=n+1}^{n+m} x_i^{b_{ij}}$ that encodes the bottom half of the matrix, C.

These have a (tropical) mutation rule:

$$Y_j' = \begin{cases} Y_k^{-1} & j=k \\ Y_j (Y_k \oplus 1)^{-b_{kj}} & j\neq k,\ b_{kj} \le 0 \\ Y_j (Y_k^{-1} \oplus 1)^{-b_{kj}} & j\neq k,\ b_{kj} \ge 0 \end{cases}$$

where $\prod_i x_i^{a_i} \oplus \prod_i x_i^{b_i} = \prod_i x_i^{\min(a_i,b_i)}$

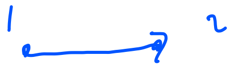
We can then compute the exchange graph for a given cluster algebra in terms of the Y_j, without specifying C, and see that the exchange graph is independent of the y-variables.

Principal Coeff.

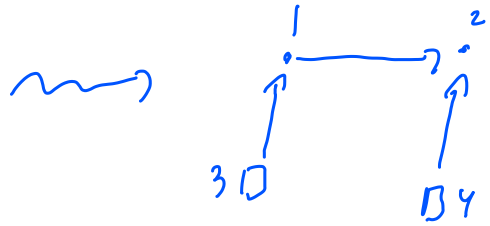
We pick an extended quiver that "plays nice" with other cluster algebra data.

Consider an initial seed $((x_1, \dots, x_n), B_0)$ with B_0 a skew-symmetrizable $n \times n$. This defines a cluster algebra with principal coefficients given by $\left((x_1, \dots, x_{2n}), \frac{B_0}{I}\right)$

E.g.:



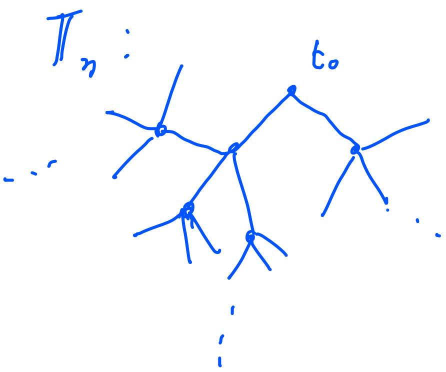
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We now establish some notation.

Let $\mathbb{T}_n$ be the n-ary tree with root labelled t_0



Label the edges s.t each vertex is incident to exactly n edges labeled $\{1, 2, \dots, n\}$.

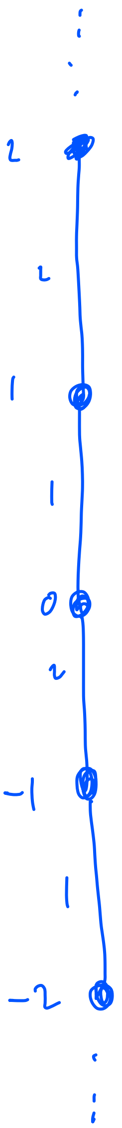
We put a seed at each vertex. The initial seed goes at t_0 and each edge represents a mutation:

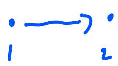
$$((x_1,\dots,x_k,\dots,x_{n+m}), B) \;\overset{k}{\text{————}}\; ((x_1,\dots,x_k',\dots,x_{n+m}), \mu_k(B))$$

The exchange graph is obtained from $\mathbb{T}_n$ by identifying certain vertices.

We will denote the cluster variables at vertex t by $x_i^{(t)}$ and the exchange matrix by $B^{(t)} = \left(\begin{array}{c} B_0^{(t)} \\ \hline C^{(t)} \end{array}\right) = \left(b_{ij}^{(t)}\right)_{\substack{1\le i\le m+n \\ 1\le j\le n}}$

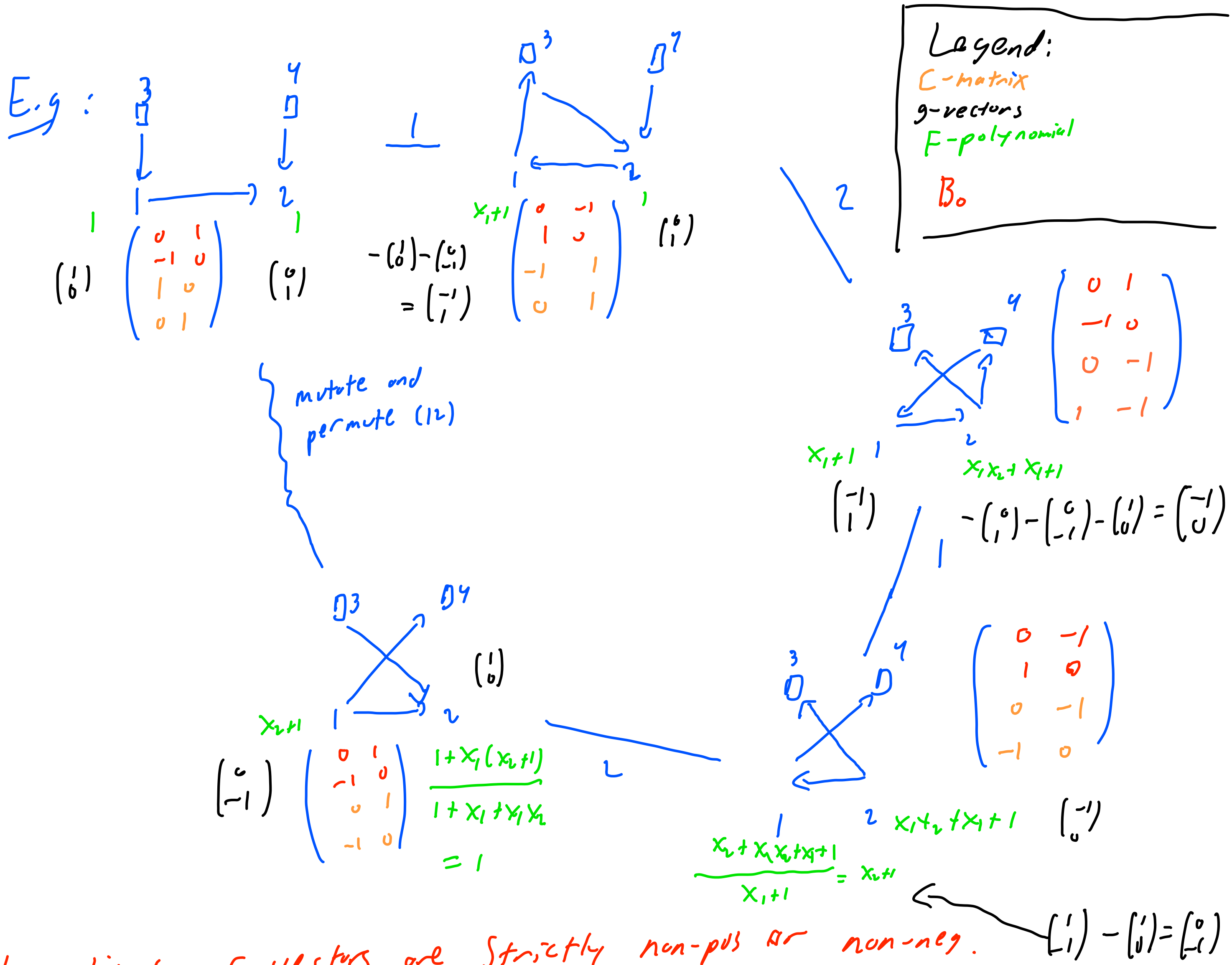
E.g. $\mathbb{T}_2$



	Quivers	B	Cluster variables
$\vdots$		$\vdots$	
1		$B^{(1)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$x_1^{(1)} = \frac{x_2+1}{x_1}$, $x_2^{(1)} = x_2$
0		$B^{(0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$x_1^{(0)} = x_1$, $x_2^{(0)} = x_2$
-1		$B^{(-1)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$x_1^{(-1)} = x_1$, $x_2^{(-1)} = \frac{x_1+1}{x_2}$
-2		$B^{(-2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$x_1^{(-2)} = \frac{x_1x_2+x_1+1}{x_1x_2}$, $x_2^{(-2)} = \frac{x_1+1}{x_2}$
$\vdots$		$\vdots$	

Def: Given an initial seed $(x^{(0)}, B_0^{(0)})$, construct the cluster algebra with principal coefficients, with initial seed $\left((x_1^{(0)}, \dots, x_{2n}^{(0)}), \left(\frac{B^{(0)}}{I} \right) \right)$. Then, the columns of $C^{(t)}$ are the <u>C-vectors</u> of t. Denote them $c_1^{(t)}, \dots, c_\ell^{(t)}$

E.g:



Observation: C-vectors are strictly non-pos or non-neg.

Thm [GHKK]: C-vectors are <u>sign-coherent</u>; all entries are non-pos or all entries are nonneg.

Conjectured by Fomin-Zelevinsky

Rmk: One way to show this is by additive categorification:
Up to sign, C vectors are dim'n vectors of bricks.

C-vectors have a nice mutation rule: If we mutate at k
$t \overset{k}{—} t'$:

$$c_\ell^{(t')} = \begin{cases} -c_k^{(t)} & \text{if } \ell = k \\ c_\ell^{(t)} + \max(b_{k,\ell}^{(t)}, 0)\, c_k^{(t)} & \text{if } c_k^{(t)} \geq 0 \\ c_\ell^{(t)} + \max(-b_{k,\ell}^{(t)}, 0)\, c_k^{(t)} & \text{if } c_k^{(t)} \leq 0 \end{cases}$$

These are closely related to g-vectors. These also have a nice description from principal coefficients and a mutation rule, which I will come back to if time permits, but they are columns of $G^{(t)}$ where

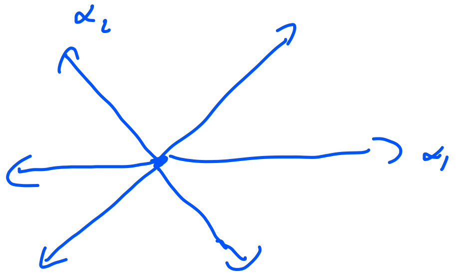
$$G^{(t)} D C^{(t)} = D,$$ D the positive diagonal matrix s.t. $(DB_0)^T = -DB_0$.

Rmk: C-vectors relate to root vectors:

For our cluster algebra on the A_2-quiver, we have 6 C-vectors:

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}$$

Consider the A_2 root system w/ simple roots α_1, α_2.



Here a bijection

$$\{c\text{-vectors}\} \longleftrightarrow \{\text{roots}\}$$
$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto a\alpha_1 + b\alpha_2 .$$

This holds for any cluster-finite cluster algebra

For each of these, can take for the initial quiver an arbitrary orientation of a Dynkin diagram in the Cartan-Killing classification of complex simple Lie algebras.

F-Polynomials

A priori, If $\mathcal{A}$ is the cluster algebra with initial seed $((x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}), B)$,

Then $\mathcal{A} \subseteq \mathbb{C}(x_1, \dots, x_{n+m})$

2 weeks ago, we saw the Laurent pheonomenon

$$\Rightarrow \mathcal{A} \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_{n+m}^{\pm 1}]$$

Even stronger (called the Sharpened Laurent Phenomenon):

Thm [FZ] $\mathcal{A} \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, x_{n+1}, \dots, x_{n+m}]$

E.g: Check this in A_2 case, worked out above.

Def: Consider a cluster alg. w/ principal coeff.

Let $X_i^{(t)}$ be the Laurent polynomial in the initial seed variables giving the i^{th} coordinate in seed t of the cluster algebra.

Define $F_i^{(t)} = X_i^{(t)}(1, 1, \ldots, 1, X_{n+1}, \ldots, X_{2n})$, $i \in [n]$.

This is a polynomial by the sharpened Laurent Phenomenon.
Call these F-polynomials.

Thm [FZ]: F-polynomials and g-vectors (dually, c-vectors) determine all cluster variables from the initial seed

$$X_\ell^{(t)} = \frac{F_\ell^{(t)}(Y_1, \ldots, Y_n)}{F_{\ell,trop}^{(t)}(\hat{Y}_1, \ldots, \hat{Y}_n)} X_1^{g_{\ell,1}^{(t)}} \cdots X_n^{g_{\ell,n}^{(t)}},$$

where $g_\ell^{(t)}$ is the g-vector for $X_\ell^{(t)}$,

$$Y_j := \prod_{i=n+1}^{n+m} X_i^{b_{ij}}, \qquad \hat{Y}_j = Y_j \cdot \prod_{i=1}^{n} X_i^{b_{ij}^{(0)}} \quad \text{and}$$

$F_{\ell,trop}^{(t)}$ is $F_\ell^{(t)}$ with each $+$ replaced by $\oplus$.

Thm The F-polynomials mutate as follows:

together with the recurrence relations

$$(5.2) \qquad F_{\ell;t'} = F_{\ell;t} \quad \text{for } \ell \neq k;$$

$$(5.3) \qquad F_{k;t'} = \frac{\prod_{j=1}^n y_j^{[b_{n+j,k}^t]_+} \prod_{i=1}^n F_{i;t}^{[b_{ik}^t]_+} + \prod_{j=1}^n y_j^{[-b_{n+j,k}^t]_+} \prod_{i=1}^n F_{i;t}^{[-b_{ik}^t]_+}}{F_{k;t}},$$

where $[a]_+ := \max(a, 0)$, and $y_j \equiv x_{j+n}$.

E.g.: See above, in green.

Extra: g-vectors

In a cluster algebra with principal coefficients, all cluster variables are homogenous w.r.t the $\mathbb{Z}^n$ grading

$$x_i \longmapsto e_i$$

$$x_{n+i} \longmapsto -B_0^{(0)} e_i$$

They mutate according to

together with the recurrence relations

(6.11) $$\mathbf{g}_{\ell;t'} = \mathbf{g}_{\ell;t} \quad \textit{for } \ell \neq k;$$

(6.12) $$\mathbf{g}_{k;t'} = -\mathbf{g}_{k;t} + \sum_{i=1}^{n} [b_{ik}^t]_+ \mathbf{g}_{i;t} - \sum_{j=1}^{n} [b_{n+j,k}^t]_+ \mathbf{b}_j^0,$$

where b_j^0 is the j^{th} col. of $B_0^{(0)}$

At first glance, it is highly nontrivial that these should be at all related to the c-vectors, which are the columns of the bottom half of the exchange matrix.