final

math133, linear algebra and geometry
summer 2022

Justify all your claims rigorously. Every question is worth 10 points and this exam is worth 55% of the grade for math133. Allotted time is three hours.

Throughout this exam, let $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

1. Using the Gauss-Jordan algorithm, solve the following linear system of equations :

$$\begin{cases} x + 4y - z & = 10 \\ x + 3y + z & = 10 \\ 2x + 2y + 4z & = 14 \end{cases}$$

2. Consider the following 3 matrices :

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ 4 & 1 & -2 \\ 1 & 2 & 5 \end{pmatrix} \qquad C = \begin{pmatrix} -2 & 2 & 5 \\ -5 & 0 & 1 \\ -3 & -1 & 0 \end{pmatrix}$$

Compute $AB + 3C$.

3. Consider the following matrix :

$$A = \begin{pmatrix} 4 & 5 & 6 \\ 4 & 9 & 13 \\ 4 & 7 & 9 \end{pmatrix}$$

- **a.** Find a product of elementary matrices B such that BA is upper triangular.
- **b.** Compute the determinant of A.

4. Let U, V be two $\mathbb{F}$-vector spaces and let $T : U \to V$ be a linear map. Show that $\operatorname{Ker} T$ is a subspace of U.
Warning : Do not cite this exact result that was proven in class to prove the result here.

5. Let U, V be two $\mathbb{F}$-vector spaces and let $T : U \to V$ be a linear map. Suppose there exists a linear map $S : V \to U$ such that $S \circ T = \mathrm{id}_U$. Show that T is injective.

6. Consider $\mathbb{C}$ as an $\mathbb{R}$-vector space. Fix the basis $\mathcal{B} = \{1, i\}$. Let $w = a + bi \in \mathbb{C}$ and define $T : \mathbb{C} \to \mathbb{C}$ by $T(z) = wz$.

- **a.** Show that T is a linear map.
- **b.** Compute $\underset{\mathcal{B} \leftarrow \mathcal{B}}{[T]}$ and compute its determinant.

7. Let $v_1, \dots, v_n \in \mathbb{R}^n$ be non-zero vectors such that they are pairwise orthogonal, i.e. for $i \neq j$, $v_i \cdot v_j = 0$. Show that $v_1, \dots, v_n$ is a basis of $\mathbb{R}^n$.

8. Let $\mathrm{Tr} : \mathbf{M}_{n \times n}(\mathbb{F}) \to \mathbb{F}$ be the map defined by

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii} \qquad A = (a_{ij}).$$

- **a.** Show that Tr is a linear map.
- **b.** Compute $\dim(\mathrm{Ker}\,\mathrm{Tr})$.
- **c.** By a direct computation, show that for $A, B \in \mathbf{M}_{n \times n}(\mathbb{F})$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.