Final, solutions

1. The augmented matrix of the system is:

$$\left(\begin{array}{ccc|c} 1 & 4 & -1 & 10 \\ 1 & 3 & 1 & 10 \\ 2 & 2 & 4 & 14 \end{array}\right) \xrightarrow[\frac{1}{2}R_3 \to R_3]{R_1 - R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 10 \\ 0 & 1 & -2 & 0 \\ 1 & 1 & 2 & 7 \end{array}\right) \xrightarrow{R_1 - R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 10 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -3 & 3 \end{array}\right) \xrightarrow{R_2 - 1/3 R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 10 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

and it follows that $z=1$, $y-2z=0 \Leftrightarrow y=2$ and $x+4y-z=10 \Leftrightarrow x=10-4\cdot 2+1=3$.

2. We have

$$AB+3C=\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ 4 & 1 & -2 \\ 1 & 2 & 5 \end{pmatrix} + 3\begin{pmatrix} -2 & 2 & 5 \\ -5 & 0 & 1 \\ -3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -5 & -14 \\ 15 & 1 & -2 \\ 9 & 4 & 1 \end{pmatrix} + \begin{pmatrix} -6 & 6 & 15 \\ -15 & 0 & 3 \\ -9 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

3.a. We have

$$\underset{\text{row add } 2,1}{\overset{E_4}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}} \underset{\text{row scale}}{\overset{E_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}} \cdot \underset{\text{row add } 3,1}{\overset{E_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}} \cdot \underset{\text{row scale}}{\overset{E_1}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}} \underbrace{\begin{pmatrix} 4 & 5 & 6 \\ 4 & 9 & 13 \\ 4 & 7 & 9 \end{pmatrix}}_{A} = \begin{pmatrix} 4 & 5 & 6 \\ 0 & -4 & -7 \\ 0 & -2 & -3 \end{pmatrix}$$

$$\underset{\text{row add } 3,2}{\overset{E_6}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}} \underset{\text{row scale}}{\overset{E_5}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}} \cdot \begin{pmatrix} 4 & 5 & 6 \\ 0 & -4 & -7 \\ 0 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 0 & -4 & -7 \\ 0 & 0 & -1 \end{pmatrix}$$

and it follows that $B=E_6E_5E_4E_3E_2E_1$ is such that BA is upper triangular.

b. We know that $\det(E_2)=\det(E_4)=\det(E_6)=1$ and $\det(E_1)=-1$, $\det(E_3)=-1$, $\det(E_5)=-2$. Thus, $\det B=-2$ and we have

$$\det(A)=\frac{\det(BA)}{\det(B)}=\frac{4\cdot(-4)\cdot(-1)}{-2}=-8.$$

4. We check that $\ker T$ respects the three conditions to be a subspace.

1) Since $T(0)=0$ (since $T(0)=T(0+0)=T(0)+T(0)$ by linearity), $0\in\ker T$.

2) Let $u_1,u_2\in\ker T$. Then, $T(u_1)=T(u_2)=0$ and $T(u_1+u_2)=T(u_1)+T(u_2)=0+0=0$ since T is linear $\Rightarrow u_1+u_2\in\ker T$.

3) Let $u\in\ker T$ and $\lambda\in\mathbb{F}$. Then, $T(u)=0$ and $T(\lambda u)=\lambda\cdot T(u)=\lambda\cdot 0=0$ since T is linear $\Rightarrow \lambda u\in\ker T$.

Thus, $\ker T$ is a subspace by definition.

5. Let $u_1, u_2 \in U$ with $T(u_1) = T(u_2)$. Then, $u_1 = id_U(u_1) = (S \circ T)(u_1) = S(T(u_1)) = S(T(u_2)) = (S \circ T)(u_2) = id_U(u_2) = u_2$ which shows T is injective.

6. a. Let $z_1, z_2 \in \mathbb{C}$ and $\lambda \in \mathbb{R}$. Then, $T(\lambda z_1 + z_2) = w(\lambda z_1 + z_2) = w\lambda z_1 + wz_2 = \lambda w z_1 + wz_2 = \lambda T(z_1) + T(z_2)$, which shows T is linear.

b. We have $T(1) = w \cdot 1 = w = a + bi$ and $T(i) = w \cdot i = (a+bi) \cdot i = -b + ai$. Consequently,

$$\underset{B \leftarrow B}{[T]} = \begin{pmatrix} \top & \top \\ [T(1)]_B & [T(i)]_B \\ \bot & \bot \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and it follows that $\det\left(\underset{B \leftarrow B}{[T]}\right) = a^2 - b(-b) = a^2 + b^2 = |w|^2$.

7. Since $\dim \mathbb{R}^n = n$, we only need to show $v_1, \dots, v_n$ are linearly independent. Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be such that $\sum \lambda_i v_i = 0$. Then, for $1 \leq j \leq n$,

$$0 = v_j \cdot \underbrace{(\lambda_1 v_1 + \dots + \lambda_n v_n)}_{=0} = \lambda_1 (v_j \cdot v_1) + \lambda_2 (v_j \cdot v_2) + \dots + \lambda_n (v_j \cdot v_n) = \lambda_j (v_j \cdot v_j) = \lambda_j \cdot \|v_j\|^2$$

$\Rightarrow \lambda_j \cdot \|v_j\|^2 = 0 \Rightarrow \lambda_j = 0$ since $\|v_j\|^2 \neq 0$ (v_j is a non-zero vector). Thus, $\lambda_1 = \dots = \lambda_n = 0 \Rightarrow$ the vectors are L.I. $\Rightarrow$ They form a basis.

8. a. Let $A, B \in M_{n \times n}(\mathbb{F})$ and let $\lambda \in \mathbb{F}$. Then, if $A = (a_{ij})$ and $B = (b_{ij})$,

$$Tr(\lambda A + B) = Tr((\lambda a_{ij} + b_{ij})) = \sum_{i=1}^{n} (\lambda a_{ii} + b_{ii}) = \lambda \cdot \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \lambda Tr(A) + Tr(B).$$

which shows Tr is linear.

b. We know by quiz 3/4 that $\dim(M_{n \times n}(\mathbb{F})) = n^2$. Moreover, $Tr(I_n) = \sum_{i=1}^{n} 1 = n \neq 0$. Thus, $\dim(Im(Tr)) = 1$ since $Im\, Tr \subseteq \mathbb{F} \Rightarrow 0 \leq \dim(Im\, Tr) \leq \dim \mathbb{F} = 1$ and it follows that

$$\dim(M_{n \times n}(\mathbb{F})) = \dim(\ker Tr) + \dim(Im\, Tr) \Rightarrow \dim \ker Tr = n^2 - 1.$$

c. Let $A, B \in M_{n \times n}(\mathbb{F})$ with $A = (a_{ij})$ and $B = (b_{ij})$. Then, $AB = (\sum_{k=1}^{n} a_{ik} b_{kj})$, $BA = (\sum_{k=1}^{n} b_{ik} a_{kj})$ which implies

$$Tr(AB) = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{ki} \right) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} a_{ik} b_{ki} \right) \underset{\substack{\text{Re-label} \\ i \mapsto k,\, k \mapsto i}}{=} \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ki} b_{ik} \right) = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} b_{ik} a_{ki} \right) = Tr(BA).$$