midterm

math133, linear algebra and geometry

summer 2022

Justify all your claims rigorously. Every question is worth 10 points and this exam is worth 35% of the grade for math133. Allotted time is two hours.

Throughout this exam, let $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

1. Let $z \in \mathbb{C}$.

a. Show that $\mathrm{Re}(z) = 0$ is equivalent to $z + \overline{z} = 0$ and show that $\mathrm{Im}(z) = 0$ is equivalent to $z - \overline{z} = 0$.

b. Show that if there exists $n \in \mathbb{Z}_{\geq 1}$ such that $z^n = 1$, then $|z| = 1$.

2. Let $c \in \mathbb{R}$. When is

$$U = \Big\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ ; \ cx_1 + 2x_2 - x_4 = 0 \text{ and } x_3 - x_4 + 1 = c\Big\}$$

a subspace of $\mathbb{R}^4$? When is U not a subspace of $\mathbb{R}^4$?

3. Let V be an $\mathbb{F}$-vector space. Show that if $U, W \subset V$ are subspaces, then $U \cap W$ is a subspace of V.

4. Let V be an $\mathbb{F}$-vector space and let $v_1, \ldots, v_m \in V$. Show that if

$$\mathrm{span}_{\mathbb{F}}\{v_1, v_2, \ldots, v_m\} = \mathrm{span}_{\mathbb{F}}\{v_2, v_3, \ldots, v_m\},$$

then $v_1, \ldots, v_m$ are linearly dependent.

5. Let $z \in \mathbb{C}$. Show that if $\mathrm{Re}(z) \neq 0$ and $\mathrm{Im}(z) \neq 0$, then $\mathrm{span}_{\mathbb{R}}\{z, \overline{z}\} = \mathbb{C}$.

6. Show that the vectors $(a, b), (c, d) \in \mathbb{R}^2$ are linearly independent if and only if $ad - bc \neq 0$.

7. Give a basis of the subspace

$$U = \big\{p(X) \in \mathbb{R}[X] \ ; \ \deg\big(p(X)\big) \leq 3\big\} \subset \mathbb{R}[X]$$

where every polynomial of this basis has degree equal to 3 and show that it is in fact a basis.