math 133 . midterm solutions

1. a. Let $z=a+bi$. Then, we have $z+\bar{z}=(a+bi)+(a-bi)=2a$ and $2a=0 \Leftrightarrow a=0 \Leftrightarrow Re(z)=0$. Moreover,
$z-\bar{z}=(a+bi)-(a-bi)=2bi$ and $2bi=0 \Leftrightarrow b=0 \Leftrightarrow Im(z)=0$.

   b. Since the norm is multiplicative (i.e. $|z\cdot w|=|z|\cdot|w| \;\forall z,w\in\mathbb{C}$), if $z\in\mathbb{C}$ such that $z^n=1$, we have that
$1=|z^n|=|\underbrace{z\cdot z\cdot\ldots\cdot z}_{n\text{-times}}|=\underbrace{|z|\cdot|z|\cdot\ldots\cdot|z|}_{n\text{-times}}=|z|^n$. But, $|z|\in\mathbb{R}_{\geq 0}$ and $|z|^n=1 \Rightarrow |z|=1$.

2. If $c\neq 1$, then $0=(0,0,0,0)\notin U$ since $0-0+1=1\neq c$. Thus, U is <u>not</u> a subspace if $c\neq 1$.
If $c=1$, we show that U is a subspace of $\mathbb{R}^4$ We verify the three necessary conditions for U to be a subspace:
   1) $0=(0,0,0,0)\in U$ since $1\cdot 0+2\cdot 0-0=0$ and $0-0+1=1$.
   2) If $u,v\in U$, then $u=(x_1,x_2,x_3,x_4)$ and $v=(y_1,y_2,y_3,y_4)$ where $x_1+2x_2-x_4=0$, $x_3-x_4+1=1$, $y_1+2y_2-y_4=0$ and $y_3-y_4+1=1$. Thus, $u+v=(x_1+y_1,x_2+y_2,x_3+y_3,x_4+y_4)$ and
$$(x_1+y_1)+2(x_2+y_2)-(x_4+y_4)=(x_1+2x_2-x_4)+(y_1+2y_2-y_4)=0+0=0,$$
$$(x_3+y_3)-(x_4+y_4)+1=(x_3-x_4+1)+(y_3-y_4+1)-1=1+1-1=1,$$
which shows that $u+v\in U$.
   3) Let $\lambda\in\mathbb{R}$ and let $u\in U$, $u=(x_1,x_2,x_3,x_4)$. Then, $x_1+2x_2-x_4=0$ and $x_3-x_4+1=1$. It follows that $\lambda u=(\lambda x_1,\lambda x_2,\lambda x_3,\lambda x_4)$ and
$$\lambda x_1+2\lambda x_2-\lambda x_4=\lambda(x_1+2x_2-x_4)=\lambda\cdot 0=0$$
$$\lambda x_3-\lambda x_4+1=\lambda(x_3-x_4+1)-\lambda+1=\lambda-\lambda+1=1$$
which shows that $\lambda u\in U$.

Consequently, U is a subspace of $\mathbb{R}^4$ by definition.

3. We verify the three necessary conditions for $U\cap W$ to be a subspace:
   1) Since U,W are subspaces, $0\in U$ and $0\in W \Rightarrow 0\in U\cap W$.
   2) If $u,v\in U\cap W \Rightarrow u,v\in U$ <u>and</u> $u,v\in W \Rightarrow u+v\in U$ <u>and</u> $u+v\in W$ since both are subspaces. This implies $u+v\in U\cap W$.
   3) If $u\in U\cap W$ and $\lambda\in\mathbb{F}$, $u\in U$ <u>and</u> $u\in W \Rightarrow \lambda u\in U$ and $\lambda u\in W$ since both are subspaces. This implies $\lambda u\in U\cap W$.

Consequently, $U\cap W$ is a subspace of V by definition.

4. Since $v_1 = 1\cdot v_1 + 0\cdot v_2 + \dots + 0\cdot v_m$, $v_1 \in \mathrm{span}\{v_1, \dots, v_m\} = \mathrm{span}\{v_2, \dots, v_m\}$. Thus, $\exists \lambda_2, \dots, \lambda_m \in \mathbb{F}$ such that $v_1 = \lambda_2 v_2 + \dots + \lambda_m v_m$ which implies that $0 = (-1)v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m$. This linear combination has at least one non-zero coeff. and gives 0, which shows that $v_1, \dots, v_m$ are linearly dependent by definition.

5. Let $z = a+bi$ with $a, b \neq 0$. Then, $\frac{1}{2a}(z+\bar{z}) = 1$ and $\frac{1}{2b}(z-\bar{z}) = i$. Thus, if $w = c+di$,

$$w = c\cdot 1 + d\cdot i = c\cdot\left(\frac{1}{2a}(z+\bar{z})\right) + d\cdot\left(\frac{1}{2b}(z-\bar{z})\right) = \underbrace{\left(\frac{c}{2a}+\frac{d}{2b}\right)}_{\in\mathbb{R}} z + \underbrace{\left(\frac{c}{2a}-\frac{d}{2b}\right)}_{\in\mathbb{R}} \bar{z} \in \mathrm{span}_{\mathbb{R}}\{z, \bar{z}\}$$

which shows the desired result.

6. If $ad-bc \neq 0$, let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_1(a,b) + \lambda_2(c,d) = (0,0)$. Then

$$\begin{cases} \lambda_1 a + \lambda_2 c = 0 & (1) \\ \lambda_1 b + \lambda_2 d = 0 & (2) \end{cases} \rightsquigarrow d\cdot(1) - c(2) : (\lambda_1 a + \lambda_2 c)d - (\lambda_1 b + \lambda_2 d)c = \lambda_1(ad-bc) = 0$$

which implies $\lambda_1 = 0 \Rightarrow \lambda_2 c = \lambda_2 d = 0$. If $c=d=0$, then $ad-bc=0$ which is impossible. Thus, $c\neq 0$ or $d \neq 0$. In both cases, $\lambda_2 = 0$.

If $ad-bc=0$, then one has

$$d\cdot(a,b) + (-b)\cdot(c,d) = (ad-bc, 0) = (0,0) \text{ and } (-c)(a,b) + a\cdot(c,d) = (0, ad-bc) = (0,0).$$

If one of a,b,c or d is $\neq 0$, one of the two previous linear combination has non-zero scalars, showing linear dependence.

If $a=b=c=d=0$, one has $1\cdot(a,b) + 1\cdot(c,d) = (0,0)$, showing linear dependence.

7. Take $p_1 = 1-x^3$, $p_2 = x-x^3$, $p_3 = x^2-x^3$, $p_4 = x^3$. Then, if $p \in U$, $p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and one has $p = a_0(1-x^3) + a_1(x-x^3) + a_2(x^2-x^3) + (a_0+a_1+a_2+a_3)x^3 \Rightarrow \mathrm{span}\{p_1, p_2, p_3, p_4\} = U$. Since $\dim U = 4$ (computation done in class), it follows that p_1, p_2, p_3, p_4 forms a basis (by a thm shown in class).