practice final

math133, linear algebra and geometry
summer 2022

Justify all your claims rigorously.

Throughout this exam, let $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

1. Using the Gauss-Jordan algorithm, solve the following linear system of equations :

$$\begin{cases} 2x + y - z & = -1 \\ -3x - 2y + z & = 0 \\ x + 5y + z & = 7 \end{cases}$$

2. Consider the following 3 matrices :

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & -3 \\ 3 & 2 & 1 \\ 4 & 5 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} -19 & 4 & 2 \\ -33 & -8 & 6 \\ -25 & 0 & 0 \end{pmatrix}$$

Compute $2AB + C$.

3. Consider the following matrix :

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 3 & 1 & -2 \end{pmatrix}$$

- **a.** Find a product of elementary matrices B such that BA is upper triangular.
- **b.** Compute the determinant of A.

4. Let U, V be two $\mathbb{F}$-vector spaces and let $T : U \to V$ be a linear map. Show that $\operatorname{Ker} T$ is a subspace of U.

5. Let U, V be two $\mathbb{F}$-vector spaces and let $T : U \to V$ be a linear map. Suppose there exists a linear map $S : V \to U$ such that $T \circ S = \mathrm{id}_V$. Show that T is surjective.

6. Consider $\mathbb{R}[X]$ as an $\mathbb{R}$-vector space and let $U = \{p \in \mathbb{R}[X] \ ; \ \deg p \leq 3\}$. Fix the basis $\mathcal{B} = \{1, X, X^2, X^3\}$ of U. Define $T : U \to \mathbb{R}[X]$ by

$$T(a_0 + a_1X + a_2X^2 + a_3X^3) = a_1 + 2a_2X + 3a_3X^2.$$

- **a.** Show that T is a linear map and that $\operatorname{Im} T \subset U$.
- **b.** Compute $\underset{\mathcal{B} \leftarrow \mathcal{B}}{[T]}$.

7. Let U, V be two finite dimensional $\mathbb{F}$-vector spaces and let $T : U \to V$ be a linear map. Show that if $\dim U > \dim V$, then T is not injective.

8. Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. Define $T_v : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T_v(u) = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1), \quad u = (u_1, u_2, u_3).$$

- **a.** Show that T_v is linear.
- **b.** Through a direct computation, show that T respects the following two properties :

$$T_v(u) \cdot w = T_w(v) \cdot u \qquad T_{T_w(v)}(u) = (u \cdot w)v - (u \cdot v)w$$

  for all $u, v, w \in \mathbb{R}^3$.
- **c.** Conclude that $\|T_v(u)\|^2 = \|u\|^2 \|v\|^2 \sin^2 \theta$ where θ is the angle between u and v.