Practice final, solutions (last update 31^{st} of May, 19:00)

1. Apply Gauss-Jordan:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ -3 & -2 & 1 & 0 \\ 1 & 5 & 1 & 7 \end{array}\right) \xrightarrow[\text{row } 1 - 2\text{row } 3 \to \text{row } 3]{3\text{row } 1 + 2\text{row } 2 \to \text{row } 2} \left(\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & -9 & -3 & -15 \end{array}\right) \xrightarrow{9\text{row}2 - \text{row}3 \to \text{row}3} \left(\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -6 & -12 \end{array}\right)$$

Now, propagating solutions, we get

$\Rightarrow -6z = -12 \Rightarrow z = 2$ by row 3.

$\Rightarrow -y - 2 = -3 \Rightarrow y = 1$ by row 2.

$\Rightarrow 2x + 1 - 2 = -1 \Rightarrow x = 0$ by row 1.

Thus, the solution is $x = 0, y = 1, z = 2$.

2. a) One has

$$AB = \begin{pmatrix} 10 & 8 & -1 \\ 17 & 14 & -3 \\ 13 & 10 & 0 \end{pmatrix} \Rightarrow 2AB = \begin{pmatrix} 20 & 16 & -2 \\ 39 & 28 & -6 \\ 26 & 20 & 0 \end{pmatrix} \Rightarrow 2AB + C = \begin{pmatrix} 1 & 20 & 0 \\ 1 & 20 & 0 \\ 1 & 20 & 0 \end{pmatrix}$$

3. One has

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 3 & 1 & -2 \end{pmatrix} \rightsquigarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}}_{\substack{E_2 \\ \text{row 3 scale}}} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{E_1 \\ \text{row 2 scale}}} \cdot A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & \frac{1}{2} \\ -1 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{\substack{E_4 \\ \text{row add } 3,1}} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{E_3 \\ \text{row add } 2,1}} \cdot \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1/2 \\ -1 & -1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 3/2 \\ 0 & -4/3 & 5/3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{\substack{E_6 \\ \text{row add } 3,2}} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3/4 \end{pmatrix}}_{\substack{E_5 \\ \text{row 3 scale}}} \cdot \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 3/2 \\ 0 & -4/3 & 5/3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 3/2 \\ 0 & 0 & 1/4 \end{pmatrix}$$

and $B = (E_6 E_5 E_4 E_3 E_2 E_1)$ is such that BA is upper triangular, as required.

b) We have $\det E_6 = \det E_4 = \det E_3 = 1$ and $\det E_5 = -3/4$, $\det E_2 = -1/3$, $\det E_1 = -1/2$. Also, $\det BA = 1 \cdot (-1) \cdot \frac{1}{4} = \frac{-1}{4}$ since BA is upper triangular. Thus,

$$\det A = \frac{\det(BA)}{\det(B)} = \frac{-1/4}{\frac{-3}{4} \cdot \frac{-1}{3} \cdot \frac{-1}{2}} = \frac{4 \cdot 3 \cdot 2}{4 \cdot 3} = 2$$

4. We check the three conditions for $\ker T$ to be a subspace.

1) Since $T(0) = 0$, one has $0 \in \ker T$.

2) If $u_1, u_2 \in \ker T$, then $T(u_1 + u_2) = T(u_1) + T(u_2) = 0 + 0 = 0$ since T is linear. Thus, $u_1 + u_2 \in \ker T$.

3) If $u \in \ker T$ and $\lambda \in \mathbb{F}$, then $T(\lambda u) = \lambda \cdot T(u) = \lambda \cdot 0 = 0$ since T is linear. Thus, $\lambda u \in \ker T$.

It follows that $\ker T$ is a subspace by definition.

5. Let $v \in V$. Thus, $S(v) \in U$ and $T(S(v)) = (T \circ S)(v) = id_V(v) = v \Rightarrow v \in \operatorname{Im} T$. Thus, T is surjective.

6. a. Let $p(X) = a_0 + a_1 X + a_2 X^2 + a_3 X^3$ and $q(X) = b_0 + b_1 X + b_2 X^2 + b_3 X^3$. Let $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned} T(\lambda p(X) + q(X)) &= T((\lambda a_0 + b_0) + (\lambda a_1 + b_1)X + (\lambda a_2 + b_2)X^2 + (\lambda a_3 + b_3)X^3) \\ &= (\lambda a_1 + b_1) + 2(\lambda a_2 + b_2)X + 3(\lambda a_3 + b_3)X^2 = \lambda(a_1 + 2a_2 X + 3a_3 X^2) + (b_1 + 2b_2 X + 3b_3 X^3) \\ &= \lambda T(p(X)) + T(q(X)). \end{aligned}$$

Also, $\deg(T(p(X))) = \deg(a_1 + 2a_2 X + 3a_3 X^2) \leq 3 \Rightarrow T(p(X)) \in U$.

b. We have $T(1) = 0$, $T(X) = 1$, $T(X^2) = 2X$, $T(X^3) = 3X^2$

$$\Rightarrow \underset{B \leftarrow B}{[T]} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

7. By a thm shown in class, $\dim U = \dim \ker T + \dim \operatorname{Im} T$ and since $\operatorname{Im} T$ is a subspace of V, $\dim \operatorname{Im} T \leq \dim V$ we conclude that $\dim U \leq \dim \ker T + \dim V$. But $\dim U - \dim V > 0 \Rightarrow \dim \ker T > 0$ $\Rightarrow \ker T \neq \{0\} \Rightarrow T$ is not injective.

8.a. Let $u=(u_1,u_2,u_3)$ $w=(w_1,w_2,w_3)$. Then,

$$T_v(\lambda u+w)=T_v((\lambda u_1+w_1,\ \lambda u_2+w_2,\ \lambda u_3+w_3))$$

$$=\left((\lambda u_2+w_2)v_3-(\lambda u_3+w_3)v_2,\ (\lambda u_3+w_3)v_1-(\lambda u_1+w_1)v_3,\ (\lambda u_1+w_1)v_2-(\lambda u_2+w_2)v_1\right)$$

$$=\left(\lambda u_2v_3-\lambda u_3v_2+w_2v_3-w_3v_2,\ \lambda u_3v_1-\lambda u_1v_3+w_3v_1-w_1v_3,\ \lambda u_1v_2-\lambda u_2v_1+w_1v_2-w_2v_1\right)$$

$$=\lambda\left(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1\right)+\left(w_2v_3-w_3v_2,\ w_3v_1-w_1v_3,\ w_1v_2-w_2v_1\right)$$

$$=\lambda T_v(u)+T_v(w).$$

b. $$T_v(u)\cdot w=\left(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1\right)\cdot(w_1,w_2,w_3)$$

$$=u_2v_3w_1-u_3v_2w_1+u_3v_1w_2-u_1v_3w_2+u_1v_2w_3-u_2v_1w_3$$

$$T_w(v)\cdot u=\left(v_2w_3-v_3w_2,\ v_3w_1-v_1w_3,\ v_1w_2-v_2w_1\right)\cdot(u_1,u_2,u_3)$$

$$=v_2w_3u_1-v_3w_2u_1+v_3w_1u_2-v_1w_3u_2+v_1w_2u_3-v_2w_1u_3$$

which shows the first equality.

$$T_{T_w(v)}(u)=\left(u_2(v_1w_2-v_2w_1)-u_3(v_3w_1-v_1w_3),\ u_3(v_2w_3-v_3w_2)-u_1(v_1w_2-v_2w_1),\ u_1(v_3w_1-v_1w_3)-u_2(v_2w_3-v_3w_2)\right)$$

and

$(u\cdot w)v-(u\cdot v)w$

$$=(u_1w_1+u_2w_2+u_3w_3)(v_1,v_2,v_3)-(u_1v_1+u_2v_2+u_3v_3)(w_1,w_2,w_3)$$

$$=\left(\cancel{u_1w_1v_1}+u_2w_2v_1+u_3w_3v_1,\ u_1w_1v_2+\cancel{u_2w_2v_2}+u_3w_3v_2,\ u_1w_1v_3+u_2w_2v_3+\cancel{u_3w_3v_3}\right)$$

$$-\left(\cancel{u_1v_1w_1}+u_2v_2w_1+u_3v_3w_1,\ u_1v_1w_2+\cancel{u_2v_2w_2}+u_3v_3w_2,\ u_1v_1w_3+u_2v_2w_3+\cancel{u_3v_3w_3}\right)$$

which shows the second equality.

c. One has

$$\|T_v(u)\|^2=T_v(u)\cdot T_v(u)\underset{\substack{\uparrow\\ \text{eqn 1}}}{=}T_{T_v(u)}(v)\cdot u\underset{\substack{\uparrow\\ \text{eqn 2}}}{=}\left((v\cdot v)u-(v\cdot u)v\right)\cdot u=(v\cdot v)(u\cdot u)-(u\cdot v)(u\cdot v)$$

$$=\|v\|^2\cdot\|u\|^2-\|v\|^2\|u\|^2\cos^2\theta=\|u\|^2\cdot\|v\|^2(1-\cos^2\theta)=\|u\|^2\cdot\|v\|^2\sin^2\theta.$$

which is the desired result.