practice midterm

math133, linear algebra and geometry
summer 2022

Justify all your claims rigorously.

1. Compute the following product of two polynomials :

$$(X-1)(X^5+X^4+X^3+X^2+X+1).$$

Conclude that for $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, one has

$$z^5+z^4+z^3+z^2+z+1=0.$$

2. Show that if V, W are two $\mathbb{F}$-vector spaces, then the cartesian product

$$V\times W=\{(v,w)\ ;\ v\in V, w\in W\}$$

endowed with the two operations defined by

$$(v_1,w_1)+(v_2,w_2)=(v_1+v_2,w_1+w_2) \quad \text{and} \quad \lambda(v,w)=(\lambda v,\lambda w) \quad v,v_1,v_2\in V,\ w,w_1,w_2\in W,\ \lambda\in\mathbb{F}$$

is an $\mathbb{F}$-vector space.

3. Let V be an $\mathbb{F}$-vector space. Show that if $U, W\subset V$ are subspaces, then $U\cap W$ is a subspace of V.

4. Let V be an $\mathbb{F}$-vector space, let $v_1,\dots,v_m\in V$ be linearly independent and let $w\in V$. Show that if $v_1+w,\dots,v_m+w$ are linearly dependent, then $w\in\operatorname{span}_{\mathbb{F}}\{v_1,\dots,v_m\}$. Is the converse true, namely if $w\in\operatorname{span}_{\mathbb{F}}\{v_1,\dots,v_m\}$, then $v_1+w,\dots,v_m+w$ are linearly dependent ?

5. Let $p(X), q(X)\in\mathbb{R}[X]$ such that $p(X), q(X)$ are linearly independent. When are the three polynomials $p(X)$, $q(X)$ and $p(X)\cdot q(X)$ linearly independent ? When are they linearly dependent ?

6. Give a basis of the subspace $\{(x,y,z)\in\mathbb{R}^3\ ;\ x+y+z=0\}$ and show that it is in fact a basis.

7. Let V, W be two finite dimensional vector spaces. Recall from question **2** that $V\times W$ also has a vector space structure. Show that $V\times W$ is finite dimensional and compute $\dim(V\times W)$ in terms of $\dim V$ and $\dim W$.