practice midterm. solutions

1. We have

$$(X-1)(X^5+X^4+X^3+X^2+X+1) = X^6+X^5+X^4+X^3+X^2+X-X^5-X^4-X^3-X^2-X-1 = X^6-1$$

and since $z^6=1$ (see quiz 1), by substituting in the above polynomial, we get

$$(z-1)(z^5+z^4+z^3+z^2+1) = z^6-1=0 \Rightarrow z^5+z^4+z^3+z^2+z+1=0$$

since $z-1\neq 0$.

2. We check the 7 conditions defining a vector space for $V\times W$ with the operations defined in the question:

1) $(v_1,w_1)+(v_2,w_2)=(v_1+v_2,w_1+w_2)\overset{\text{commutativity in } V \text{ and } W}{=}(v_2+v_1,w_2+w_1)=(v_2,w_2)+(v_1,w_1)$

2) $(v_1,w_1)+((v_2.w_2)+(v_3,w_3)) = (v_1,w_1)+((v_2+v_3,w_2+w_3)) = (v_1+(v_2+v_3), w_1+(w_2+w_3)) \overset{\text{associativity in } V \text{ and } W}{=} ((v_1+v_2)+v_3,(w_1+w_2)+w_3)$

$= (v_1+v_2, w_1+w_2)+(v_3,w_3) = ((v_1,w_1)+(v_2,w_2))+(v_3,w_3)$

3) $0=(\underset{0 \text{ vector in } V}{0},\underset{0 \text{ vector in } W}{0})$, $0+(v,w)=(0,0)+(v,w)=(0+v,0+w)=(v,w)$.

4) For $v\in V$, $w\in W$, we have $(-v,-w)\in V\times W$ and $(v,w)+(-v,-w)\overset{\text{By (4) in } V \text{ and } W.}{=}(v-v,w-w)=(0,0)$

5) $1\cdot(v,w)=(1\cdot v,1\cdot w)\overset{\text{since 1 acts as identity in both } V \text{ and } W.}{=}(v,w)$

6) $\lambda\cdot(\mu\cdot(v,w)) = \lambda\cdot(\mu v,\mu w) = (\lambda(\mu v),\lambda(\mu w)) \overset{\text{By (6) in } V \text{ and } W}{=} ((\lambda\mu)v,(\lambda\mu)w) = (\lambda\mu)\cdot(v,w)$

7) $\lambda((v_1,w_1)+(v_2,w_2)) = \lambda\cdot(v_1+v_2,w_1+w_2) = (\lambda(v_1+v_2),\lambda(w_1+w_2)) \underset{\text{By (7) in } V \text{ and } W}{=} (\lambda v_1+\lambda v_2, \lambda w_1+\lambda w_2) = (\lambda v_1,\lambda w_1)+(\lambda v_2,\lambda w_2)$

$= \lambda(v_1,w_1)+\lambda(v_2,w_2)$

$(\lambda+\mu)(v,w) = ((\lambda+\mu)v,(\lambda+\mu)w) \overset{\text{By (7) in } V \text{ and } W.}{=} (\lambda v+\mu v,\lambda w+\mu w) = (\lambda v,\lambda w)+(\mu v,\mu w) = \lambda(v,w)+\mu(v,w)$.

Thus, it is a vector space by definition.

3. We check the 3 defining conditions for subspaces:

1) Since U and W are subspaces, $0\in U$ and $0\in W \Rightarrow 0\in U\cap W$.

2) Let $u,v\in U\cap W$. Then $u,v\in U$ and $u,v\in W$. Consequently, $u+v\in U$ and $u+v\in W$ since they are subspaces. It follows that $u+v\in U\cap W$.

3) Let $u\in U\cap W$ and let $\lambda\in\mathbb{F}$. Then, $u\in U$ and $u\in W$. Consequently, $\lambda u\in U$ and $\lambda u\in W$ since they are subspaces. It follows that $\lambda u\in U\cap W$.

Thus, $U\cap W$ is a subspace of V.

4. Since $v_1+w,\dots,v_m+w$ are lin. dependent, $\exists \lambda_1,\dots,\lambda_m \in \mathbb{F}$ (not all equal to zero) such that $\sum_{i=1}^m \lambda_i(v_i+w)=0$. It follows that

$$0=\sum_{i=1}^m \lambda_i(v_i+w)=\sum_{i=1}^m \lambda_i v_i+(\sum_{i=1}^m \lambda_i)\cdot w.$$

If $\sum_{i=1}^m \lambda_i=0$, then the previous eqn becomes $0=\sum_{i=1}^m \lambda_i v_i$ which forces $\lambda_1=\lambda_2=\dots=\lambda_m=0$ since the v_i's are lin. independent by hypothesis. But, we know that some λ_i is $\neq 0$. Contradiction.

Thus, $(\sum_{i=1}^m \lambda_i)\neq 0$, and it follows that $w=\sum_{i=1}^m \left(\frac{-\lambda_i}{\sum_{j=1}^m \lambda_j}\right)v_i \Rightarrow w\in \operatorname{span}\{v_1,\dots,v_m\}$ as required.

The converse is false: Take $V=\mathbb{R}^2$, $\mathbb{F}=\mathbb{R}$, $m=1$, $v_1=(1,0)$ and $w=(0,0)$. Then, v_1 is a lin. independent list, $w\in\operatorname{span}\{v_1\}$ and $v_1+w=v_1$ is a lin. independent list.

5. Let $\deg(p)=m$ and $\deg(q)=n$. Since p,q are lin. independent, $p,q\neq 0 \Rightarrow p\cdot q\neq 0$. Thus, $p\cdot q$ is a non-zero polynomial of degree $m+n$. Let $p(x)=\sum_{i=0}^m a_i x^i$ and let $q(x)=\sum_{i=0}^n b_i x^i$.

<u>Case 1</u>. If $m,n\geq 1$, we show that p,q,pq are linearly independent. Let $\lambda_1,\lambda_2,\lambda_3\in\mathbb{F}$ such that $\lambda_1 p+\lambda_2 q+\lambda_3 pq=0$. Then, the term of degree $m+n$ in $\lambda_1 p+\lambda_2 q+\lambda_3 pq$ is $\lambda_3 a_m b_n x^{m+n}$ since $n,m\geq 1$. Thus, one needs to have $\lambda_3 a_m b_n=0$ $\Rightarrow \lambda_3=0$ since $a_m\neq 0$ and $b_n\neq 0$ (otherwise, $\deg(p)$ or $\deg(q)$ would be smaller). We now have $\lambda_1 p+\lambda_2 q=0$ which implies $\lambda_1=\lambda_2=0$ since p and q are lin independent by hypothesis. It follows that $\lambda_1=\lambda_2=\lambda_3=0 \Rightarrow p,q,pq$ lin indp.

<u>Case 2</u>. If $m=0$, take $0\cdot p+a_0\cdot q+(-1)\cdot pq=a_0\cdot q-a_0\cdot q=0 \Rightarrow$ lin. dep.

<u>Case 3</u>. If $n=0$, take $b_0\cdot p+0\cdot q+(-1)\cdot pq=b_0\cdot p-b_0\cdot p=0 \Rightarrow$ lin dep

6. Take $u_1=(1,0,-1)$ and $u_2=(0,1,-1)$. Clearly, $u_1,u_2\in U$.

Are they lin indp.? Let $\lambda_1,\lambda_2\in\mathbb{R}$ such that $\lambda_1u_1+\lambda_2u_2=0$. Then, $\lambda_1u_1+\lambda_2u_2=(\lambda_1,0,-\lambda_1)+(0,\lambda_2,-\lambda_2)=(\lambda_1,\lambda_2,-\lambda_1-\lambda_2)=(0,0,0)$ which implies $\lambda_1=\lambda_2=0$.

Do they span U? Let $(x,y,z)\in U$. Then, $x+y+z=0$. Moreover, $x\cdot u_1+y\cdot u_2=(x,0,-x)+(0,y,-y)=(x,y,-x-y)=(x,y,z)$.

By definition, u_1,u_2 form a basis.

7. Let $v_1,\dots,v_m$ be a basis of V and $w_1,\dots,w_n$ be a basis of W. Consider $(v_1,0),\dots,(v_m,0),(0,w_1),\dots,(0,w_n)\in V\times W$. We show it is a basis of $V\times W$.

Are they lin. indp.? Let $\lambda_1,\dots,\lambda_m,\mu_1,\dots,\mu_n\in\mathbb{F}$ be such that $\left(\sum_{i=1}^m \lambda_i(v_i,0)\right)+\left(\sum_{i=1}^n (0,w_i)\right)=0$. Then, by def of scalar mult. and add in $V\times W$, we get $(0,0)=(\sum_{i=1}^m \lambda_i v_i, \sum_{i=1}^n \mu_i w_i) \Rightarrow \sum_{i=1}^m \lambda_i v_i=0$ and $\sum_{i=1}^n \mu_i w_i=0$. But, the v_i's and w_i's are lin. indp $\Rightarrow \lambda_i = 0 \ \forall i$ and $w_i = 0 \ \forall i$. $\Rightarrow$ vectors are lin. indp.

Do they span? Let $(v,w) \in V \times W$. Since the v_i's form a basis of V, $\exists \lambda_1, \dots, \lambda_m$ s.t. $\sum_{i=1}^{m} \lambda_i v_i = v$. Since the w_i's form a basis of W, $\exists \mu_1, \dots, \mu_n$ s.t. $\sum_{i=1}^{n} \mu_i w_i = w$. Consequently, $(v,w) = \sum_{i=1}^{m} \lambda_i (v_i, 0) + \sum_{i=1}^{n} \mu_i (0, w_i)$ $\Rightarrow$ they span.

Now, since $V \times W$ admits a finite spanning set, it is finite dim'l. Also, the basis defined above has length $n+m$. We conclude that

$$\dim(V \times W) = m+n = \dim V + \dim W.$$