quiz 3/4

math133, linear algebra and geometry
summer 2022

Justify all your claims rigorously.

1. Consider the set of quaternions

$$\mathbb{H} = \{a + bi + cj + dk \; ; \; a, b, c, d \in \mathbb{R}\}$$

as an $\mathbb{R}$-vector space. Remember that the multiplication table for $\mathbb{H}$ is the following :

$\circlearrowright$	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

Consider the basis $\mathcal{B} = \{1, i, j, k\}$ for $\mathbb{H}$. Fix $q \in \mathbb{H}$ a non-zero quaternion such that $|q| = 1$.

- **a.** Show that the map $T : \mathbb{H} \to \mathbb{H}$ given by $T(p) = q \cdot p \cdot q^{-1}$ is $\mathbb{R}$-linear.
- **b.** Compute the matrix representation of T in the bases $\mathcal{B}$ and $\mathcal{B}$, namely $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} \in \mathbf{M}_{4\times 4}(\mathbb{R})$.

2. Consider $\mathbb{R}^n$ as an $\mathbb{R}$-vector space. Fix a non-zero vector $v \in \mathbb{R}^n$. As defined in tutorial 5, for $u \in \mathbb{R}^n$, denote $\mathrm{proj}_v(u) = \frac{u \cdot v}{\|v\|^2} v$.

- **a.** Show that $\mathrm{proj}_v : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map.
- **b.** Compute the matrix representation of proj_v in the canonical basis of $\mathbb{R}^n$.
- **c.** Compute $\dim \operatorname{Ker} \mathrm{proj}_v$.

3. Consider $\mathbf{M}_{m\times n}(\mathbb{F})$ as an $\mathbb{F}$-vector space. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let E_{ij} be the matrix having every coefficient equal to zero except at position i, j (ith row and jth column) where the coefficient is equal to 1. For example, if $m = 3$, $n = 4$, one has

$$E_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- **a.** Show that the elements E_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$ form a basis of $\mathbf{M}_{m\times n}(\mathbb{F})$ and compute $\dim \mathbf{M}_{m\times n}(\mathbb{F})$.
- **b.** Suppose $m = n$. Compute $E_{ij} \cdot E_{k\ell}$ in terms of the basis given in part **a**.

4. Let V be a finite dimensional $\mathbb{F}$-vector space. Let U be a subspace of V. Show that there exists a linear map $T : V \to V$ such that $\operatorname{Ker} T = U$.