Quiz 3/4 solutions. (partial)

1. a. Let $p_1, p_2 \in \mathbb{H}$ and $\lambda \in \mathbb{R}$. One has

$$T(\lambda p_1 + p_2) = q(\lambda p_1 + p_2)q^{-1} = q(\lambda p_1)q^{-1} + q p_2 q^{-1}$$

but $q\lambda = (a+bi+cj+dk)\cdot\lambda = a\lambda + b\lambda i + c\lambda j + d\lambda k = \lambda(a+bi+cj+dk) = \lambda q$ car $\lambda \in \mathbb{R}$. Donc, on a $T(\lambda p_1 + p_2) = \lambda q p_1 q^{-1} + q p_2 q^{-1} = \lambda T(p_1) + T(p_2)$.

b. Since $q^{-1} = \bar{q}/|q|^2 = \bar{q}$, a direct computation shows

$T(1) = 1$

$T(i) = (a^2+b^2-c^2-d^2)i + (2bc+2ad)j + (2bd-2ac)k$

$T(j) = (2bc-2ad)i + (a^2+c^2-b^2-d^2)j + (2ab+2cd)k$

$T(k) = (2ac+2bd)i + (2cd-2ab) + (a^2+d^2-b^2-c^2)k$

and we thus have

$$[T]_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 0 & 2bc+2ad & a^2+c^2-b^2-d^2 & 2cd-2ab \\ 0 & 2bd-2ac & 2ab+2cd & a^2+d^2-b^2-c^2 \end{pmatrix}.$$

2. a. Let $u_1, u_2 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Since $(\lambda u_1 + u_2)\cdot v = \lambda(u_1 \cdot v) + (u_2 \cdot v)$, one has

$$\mathrm{proj}_v(\lambda u_1 + u_2) = \frac{(\lambda u_1 + u_2)\cdot v}{\|v\|^2}\cdot v = \frac{(\lambda(u_1\cdot v) + (u_2 \cdot v))}{\|v\|^2}\cdot v = \lambda\left(\frac{u_1\cdot v}{\|v\|^2}\cdot v\right) + \left(\frac{u_2\cdot v}{\|v\|^2}\right)\cdot v = \lambda\,\mathrm{proj}_v(u_1) + \mathrm{proj}_v(u_2)$$

b. Let $v = (v_1, \dots, v_n)$. Then,

$$\mathrm{proj}_v(e_i) = \frac{e_i \cdot v}{\|v\|^2} v = \frac{1}{\|v\|^2}\cdot\sum_{j=1}^{n} v_i v_j e_j$$

$$\Rightarrow [T]_{\mathcal{E}\leftarrow\mathcal{E}} = \frac{1}{\|v\|^2}\begin{pmatrix} v_1v_1 & v_1v_2 & \cdots & v_1v_n \\ v_1v_2 & v_2v_2 & \cdots & v_2v_n \\ \vdots & \vdots & \ddots & \\ v_1v_n & v_2v_n & \cdots & v_nv_n \end{pmatrix}$$

c. We have that $\mathrm{proj}_v(u) \in \mathrm{span}\{v\}$ $\forall u \in \mathbb{R}^n \Rightarrow 0 \le \dim(\mathrm{Im}\,\mathrm{proj}_v) \le 1$. But, since $\mathrm{proj}_v(v) = v$, it follows that $\dim \mathrm{Im}\,\mathrm{proj}_v = 1$.
Using a thm shown in class:

$$\dim \ker \mathrm{proj}_v = \dim \mathbb{R}^n - \dim \mathrm{Im}\,\mathrm{proj}_v = n-1.$$

4. Let $u_1, \dots, u_n \in U$ be a basis of U. By a result shown in class, there exists $v_1, \dots, v_k \in V$ such that $u_1, \dots, u_n, v_1, \dots, v_k$ is a basis of V. Define, for $v \in V$, $v = \sum_{i=1}^n \lambda_i u_i + \sum_{j=1}^k \mu_j v_j$, $T(v) = \sum_{j=1}^n \mu_j v_j$.
Let $v_1 = \sum_{i=1}^n \lambda_i^1 u_i + \sum_{j=1}^k \mu_j^1 v_j$, $v_2 = \sum_{i=1}^n \lambda_i^2 u_i + \sum_{j=1}^k \mu_j^2 v_j$ and let $\alpha \in \mathbb{F}$. Then,

$$T(\alpha v_1 + v_2) = T\left(\sum_{i=1}^n (\alpha \lambda_i^1 + \lambda_i^2) u_i + \sum_{j=1}^k (\alpha \mu_j^1 + \mu_j^2) v_j \right) = \sum_{j=1}^k (\alpha \mu_j^1 + \mu_j^2) v_j = \alpha \sum_{j=1}^k \mu_j^1 v_j + \sum_{j=1}^k \mu_j^2 v_j = \alpha T(v_1) + T(v_2).$$

which shows T is linear.
Now, clearly $U \subseteq \ker T$. If $v \in \ker T$, then if $v = \sum_{i=1}^n \lambda_i u_i + \sum_{j=1}^k \mu_j v_j \Rightarrow \mu_1 = \dots = \mu_k = 0 \Rightarrow v \in \mathrm{span}\{u_i\} = U$.
Thus, $\ker T = U$ as required.