MATH133 – Summer 2022 – Tutorial 2

Remember to make sure you understand what a question is asking before you answer it, and when in doubt, go back to the definitions. I strongly encourage you to attempt all problems in this tutorial, but do not worry if you get stuck!

1. *Rationals.* Let $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$. Show that if $\frac{p_1}{q_1} = \frac{p_2}{q_2}$, for any $\frac{a}{b} \in \mathbb{Q}$, $\frac{p_1}{q_1} + \frac{a}{b} = \frac{p_2}{q_2} + \frac{a}{b}$ and $\frac{p_1}{q_1} \cdot \frac{a}{b} = \frac{p_2}{q_2} \cdot \frac{a}{b}$. *Hint:* $\frac{p_1}{q_1} = \frac{p_2}{q_2}$ *if and only if* $p_1q_2 - p_2q_1 = 0$.

2. *Primes.* Show that for any prime number p, $\sqrt{p}$ is not rational. *Hint: the proof is very similar to the proof for* $\sqrt{2}$.

3. *Inverses.* Find the multiplicative inverse and the norm of the following complex numbers:

   (a) 1.

   (b) i.

   (c) $4 + 2i$.

   (d) $\pi - 4i$.

   (e) $\frac{1}{2} + \frac{7}{8}i$.

   (f) $cos(\theta) + isin(\theta)$, where θ is some fixed real number.

4. *Rotations.*

   (a) Let $z \in \mathbb{C}$. Show that if $|z| = 1$, there exists a $\theta \in [0, 2\pi)$ such that $z = \cos(\theta) + i\sin(\theta)$.

   (b) Let w, z be two complex numbers with $|w| = |z| = 1$. Compute $w \cdot z$, and give a geometric interpretation. *Hint: recall that* $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ *and* $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

5. *Triangles.* Prove Pythagoras' Theorem for $\mathbb{R}^3$, namely that the distance from the origin to a point (a, b, c) is $\sqrt{a^2 + b^2 + c^2}$.

6. *Geometry.* Define

$$||(x, y)|| := \sqrt{x^2 + y^2}$$
$$||(x, y, z)|| := \sqrt{x^2 + y^2 + z^2}$$

   Show the following:

   (a) The triangle inequality in $\mathbb{R}^2$: $||u + v|| \leq ||u|| + ||v||$.

   (b) The same inequality in $\mathbb{R}^3$.

   (c) The parallelogram law in $\mathbb{R}^2$: $2||u||^2 + 2||v||^2 = ||u+v||^2 + ||u-v||^2$. *Hint: recall the law of cosines.*

7. *Imaginary parts.* Let $a, b, c, d \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $ad - bc \neq 0$ and $cz + d \neq 0$. Show that

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = (ad-bc)\frac{\operatorname{Im}(z)}{|cz+d|^2}$$