MATH133 – Summer 2022 – Tutorial 3

For this tutorial, it is useful to have a cheat sheet to help you verify the axioms of vector spaces and subspaces (which is also great for memorizing them). I strongly encourage you to attempt all problems in this tutorial, but do not worry if you get stuck!

1. *Vector Spaces.* Prove whether or not following are $\mathbb{F}$-vector spaces for $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$

   (a) $V = \{p \in \mathbb{F}[x] : deg(p) \leq 20\}$, with the usual addition and multiplication.

   (b) $V = \{p \in \mathbb{F}[x] : deg(p) = 20\}$, with the usual addition and multiplication.

   (c) $V = \{f : \mathbb{F} \to \mathbb{F} : x \leq y \implies f(x) \leq f(y)\}$, with addition defined $(f+g)(x) := f(x) + g(x)$ and multiplication $(\lambda f)(x) := \lambda f(x)$.

   (d) $V =$ the set of all infinite sequences in $\mathbb{F}$, i.e. vectors $x = (x_0, x_1, x_2, \cdots), x_i \in \mathbb{R}$, with addition

   $$(x_0, x_1, x_2, \cdots) + (y_0, y_1, y_2, \cdots) := (x_0 + y_0, x_1 + y_1, x_2 + y_2, \cdots)$$

   and multiplication

   $$\lambda(x_0, x_1, x_2, \cdots) := (\lambda x_0, \lambda x_1, \lambda x_2, \cdots)$$

   (e) $V = \mathbb{F}$ with addition defined as $x + y := x \cdot y + y$ and the usual scalar multiplication.

   (f) $V = \mathbb{F}^X := \{f : X \to \mathbb{F}\}$, where $z = f + g$ is defined pointwise, namely $z(x) = f(x) + g(x)$, and the usual scalar multiplication.

   (g) $V = \mathbb{F}$, with addition defined as $x + y := x \cdot y$, and the usual scalar multiplication.

   (h) $(*)$ $V = \mathbb{Z}$ and $\mathbb{F} = \mathbb{Z}$

2. We know that $V = \mathbb{R}^2$ is an $\mathbb{R}$-vector space with the usual operations. Consider the subset of $\mathbb{R}^2$ given by

   $$U = \{(x, 0) \in \mathbb{R}^2 : x > 0\}$$

   and define for $u = (x, 0), v = (y, 0), \lambda \in \mathbb{R}$ the operations

   $$u + v := (xy, 0)$$
   $$\lambda \cdot u := (x^\lambda, 0)$$

   (a) Show that U with operations above is a vector space over $\mathbb{R}$.

   (b) Is U a subspace of V?

3. *Subspaces.* Prove or disprove whether the following are subspaces of the indicated vector spaces:

   (a) $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0)\} \subseteq \mathbb{R}^3$.

   (b) $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$.

   (c) $U = \{p \in \mathbb{C}[x] : p$ has no terms of odd power$\} \subseteq \mathbb{C}[x]$.

   (d) $U = \{p \in \mathbb{C}[x] : p$ has no terms of even power$\} \subseteq \mathbb{C}[x]$.

   (e) For $x_0, x_1 \in \mathbb{R}$, the set of all real polynomials p such that $p(x_0) = x_1$.

   (f) $U = \{(x, y, z) \in \mathbb{R}^3 : (x + 2y, z - x + y) = (0, 0)\} \subseteq \mathbb{R}^3$.

4. *Polynomials.* Recall that $deg(p)$ denotes the degree of a non-zero polynomial p, that is the highest power of x in p. Let $p, q \in \mathbb{F}[x]$. Compute the following in terms of $deg(p)$ and $deg(q)$:

   (a) $deg(p \cdot q)$.

   (b) $deg(p + q)$.