MATH133 – Summer 2022 – Tutorial 4

Note that some of the questions here are ones I did not have time to cover in earlier tutorials, I will cover them here in preparation for the midterm. I strongly encourage you to attempt all problems in this tutorial, but do not worry if you get stuck!

1. *Imaginary parts.* Let $a, b, c, d \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $ad - bc \neq 0$ and $cz + d \neq 0$. Show that
$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = (ad - bc)\frac{\operatorname{Im}(z)}{|cz+d|^2}$$

2. *Vector Spaces.* Prove whether or not following are $\mathbb{F}$-vector spaces for $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$
   (a) $V = \mathbb{F}^X := \{f : X \to \mathbb{F}\}$, where $z = f + g$ is defined pointwise, namely $z(x) = f(x) + g(x)$, and the usual scalar multiplication.
   (b) $V = \mathbb{F}$, with addition defined as $x + y := x \cdot y$, and the usual scalar multiplication.

3. *Subspaces.* Prove or disprove whether the following are subspaces of the indicated vector spaces:
   (a) For $x_0, x_1 \in \mathbb{R}$, $\{p \in \mathbb{R}[x] : p(x_0) = x_1\} \subseteq \mathbb{R}[x]$.
   (b) $U = \{(x, y, z) \in \mathbb{R}^3 : (x + 2y, z - x + y) = (0, 0)\} \subseteq \mathbb{R}^3$.

4. *Polynomials.* Recall that $deg(p)$ denotes the degree of a non-zero polynomial p, that is the highest power of x in p. Let $p, q \in \mathbb{F}[x]$. Compute the following in terms of $deg(p)$ and $deg(q)$:
   (a) $deg(p \cdot q)$.
   (b) $deg(p + q)$.

5. *Linear Independence.* Which of the following sets are linearly independent?
   (a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e \end{pmatrix} \right\}$
   (b) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -4 \\ 5 \end{pmatrix} \right\}$

6. Let V be an $\mathbb{F}$-vector space and let $\{v_1, v_2, \cdots, v_m\} \subseteq V$. Show that if
$$\operatorname{span}_{\mathbb{F}}\{v_1, v_2, \cdots, v_m\} = \operatorname{span}_{\mathbb{F}}\{v_2, v_3, \cdots, v_m\}$$
then $\{v_1, v_2, \cdots, v_m\} \subseteq V$ is linearly dependent.