final

math133, linear algebra and geometry
summer 2023

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Date and time of the examination	June 1st 2023, 9h00

Justify all your claims rigorously. This exam is worth 54% of the grade for math133. Question 1 is worth 5 points and questions 2,3,4,5,6,7,8 are worth 7 points. Allotted time is 3 hours.

1. (5 points) Find $x_1, x_2, x_3 \in \mathbb{R}$ satisfying the following system of equations :

$$\begin{cases} 3x_1 - x_2 + 4x_3 &= -7 \\ 2x_1 + 3x_2 - 2x_3 &= 1 \\ x_1 + x_3 &= -2 \end{cases}$$

2. (7 points) Let

$$\mathscr{B} = \{(2,-1,-1),(1,-1,-2),(-1,1,1)\} \qquad \mathscr{C} = \{(2,-3),(3,-4)\}$$

be bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively (you can assume that they indeed are bases). Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3).$$

a. Show that T is linear.

b. Is T surjective ?

c. What is the dimension of the kernel of T ?

d. Compute the matrix representation of T in bases $\mathscr{B}$ and $\mathscr{C}$, i.e. $\underset{\mathscr{C}\leftarrow\mathscr{B}}{[T]}$.

3. (7 points) Let U, V, W be $\mathbb{K}$-vector spaces and let $T : U \to V$ and $S : V \to W$ be linear maps. Show that if S is injective, then $\operatorname{Ker}(S \circ T) = \operatorname{Ker} T$.

4. (7 points) Consider the following subsets of $\mathbb{R}^4$:

$$U_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ ; \ x_1 + x_4 \geq 0\}$$
$$U_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ ; \ x_2 = 0 \text{ or } x_3 = 0\}$$
$$U_3 = \{(s, t+s, t-s, 3t+2s) \in \mathbb{R}^4 \ ; \ s, t \in \mathbb{R}\}$$

Determine which subsets are subspaces and which are not. Justify your answer.

5. (7 points) Let $V = \mathbb{R}[x]$ be the $\mathbb{R}$-vector space of polynomials with coefficients in $\mathbb{R}$. Consider the map $T : \mathbb{R}[x] \to \mathbb{R}[x]$ given by

$$T(p(x)) = x^2 \cdot p(x).$$

a. Show that T is a linear map.

b. What is the kernel of T?

c. What is the image of T?

6. (7 points) Let U,V be $\mathbb{K}$-vector spaces and let $u_1, \dots, u_k \in U$ be linearly independent vectors. Show that if $T : U \to V$ is an injective linear map, then $T(u_1), \dots, T(u_k) \in V$ are linearly independent.

7. (7 points) Let V be a $\mathbb{K}$-vector space and consider $v_1, v_2, v_3 \in V$ such that $v_1 + v_2 + v_3 = 0$. Show that

$$\text{span}_{\mathbb{K}}\{v_1, v_2\} = \text{span}_{\mathbb{K}}\{v_2, v_3\}.$$

8. (7 points) Let U be a subspace of $\mathbb{R}^n$.

a. Show that

$$W = \left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \; ; \; x_1y_1 + x_2y_2 + \cdots + x_ny_n = 0 \text{ for all } (y_1, y_2, \dots, y_n) \in U\right\}$$

is a subspace of $\mathbb{R}^n$.

b. Assuming that $n = 3$ and $U = \{(x, y, z) \in \mathbb{R}^3 \; ; \; x + y + z = 0\}$, find a basis of W.