Final solutions

1. By a direct computation:

$$\left(\begin{array}{ccc|c} 3 & -1 & 4 & -7 \\ 2 & 3 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{array}\right) \xrightarrow[3L_3 - L_1 \mapsto L_3]{3L_2 - 2L_1 \mapsto L_2} \left(\begin{array}{ccc|c} 3 & -1 & 4 & -7 \\ 0 & 11 & -14 & 17 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{L_2 - 11L_3 \mapsto L_3} \left(\begin{array}{ccc|c} 3 & -1 & 4 & -7 \\ 0 & 11 & -14 & 17 \\ 0 & 0 & -3 & 6 \end{array}\right)$$

$$\Rightarrow x_3 = -2, \quad x_2 = \frac{1}{11}(17 + 14\cdot(-2)) = \frac{1}{11}(17-28) = -1 \quad x_1 = \frac{1}{3}(-7-1+8) = 0$$

2. a. Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned} T(\lambda x + y) = T(\lambda x_1 + y_1, \lambda x_2 + y_2, \lambda x_3 + y_3) &= ((\lambda x_1 + y_1) + (\lambda x_2 + y_2), (\lambda x_2 + y_2) - (\lambda x_3 + y_3)) \\ &= (\lambda(x_1 + x_2) + (y_1 + y_2), \lambda(x_2 - x_3) + (y_2 - y_3)) \\ &= \lambda(x_1 + x_2, x_2 - x_3) + (y_1 + y_2, y_2 - y_3) = \lambda T(x) + T(y) \end{aligned}$$

which shows T is linear by definition.

b. Yes. One has

$$T(2,-1,-1) = (2-1, -1-(-1)) = (1,0)$$
$$T(1,-1,-2) = (1-1, -1-(-2)) = (0,1)$$

and if $(x,y) \in \mathbb{R}^2$, then

$$T(x(2,-1,-1) + y(1,-1,-2)) = xT(2,-1,-1) + yT(1,-1,-2) = x(1,0) + y(0,1) = (x,y)$$

$\Rightarrow T$ is surjective.

c. One has that $\dim \operatorname{Im} T = 2$ and $\dim \mathbb{R}^3 = 3 \Rightarrow \dim \operatorname{Ker} T = 1$ since $\dim \mathbb{R}^3 - \dim \ker T = \dim \operatorname{Im} T$.

d. Let $u_1 = (2,-1,-1)$, $u_2 = (1,-1,-2)$, $u_3 = (-1,1,1)$ and $v_1 = (2,-3)$, $v_2 = (3,-4)$. Then,

$$T(u_1) = (1,0) \quad T(u_2) = (0,1) \quad T(u_3) = (0,0)$$

We solve for $(x,y) = \lambda_1(2,-3) + \lambda_2(3,-4)$. We have

$$\left(\begin{array}{cc|c} 2 & 3 & x \\ -3 & -4 & y \end{array}\right) \xrightarrow{3L_1 + 2L_2 \mapsto L_2} \left(\begin{array}{cc|c} 2 & 3 & x \\ 0 & 1 & 3x+2y \end{array}\right)$$

$\Rightarrow \lambda_2 = 3x+2y$ and $\lambda_1 = \frac{1}{2}(x - 3(3x+2y)) = -4x-3y$. Thus,

$$\begin{aligned} x=1,\ y=0 &: -4(2,-3)+3(3,-4) = (1,0) \\ x=0,\ y=1 &: -3(2,-3)+2(3,-4) = (0,1) \end{aligned} \quad \Rightarrow \quad \begin{aligned} T(u_1) &= -4v_1+3v_2 \\ T(u_2) &= -3v_1+2v_2 \end{aligned}$$

and it follows that

$$[T(u_1)]_e = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad [T(u_2)]_e = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad [T(u_3)]_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underset{e \leftarrow \mathcal{B}}{[T]} = \begin{pmatrix} -4 & -3 & 0 \\ 3 & 2 & 0 \end{pmatrix}.$$

3. If $x \in \operatorname{Ker} T$, then $(S \circ T)(x) = S(T(x)) \underset{\text{since } x \in \operatorname{Ker} T}{=} S(0) \underset{\text{since } S \text{ is linear}}{=} 0 \Rightarrow x \in \operatorname{Ker}(S \circ T)$

If $x \in \operatorname{Ker} S \circ T$, then $(S \circ T)(x) = 0 \Rightarrow S(T(x)) = 0 \underset{\text{since } S \text{ is injective} \Leftrightarrow \operatorname{Ker} S = \{0\}.}{\Rightarrow} T(x) = 0 \Rightarrow x \in \operatorname{Ker} T.$

4\. a. No, it is not a subspace since $(1,0,0,0) \in U_1$ but $-1 \cdot (1,0,0,0) = (-1,0,0,0) \notin U_1$ (not closed under scalar multiplication).

b. No, it is not a subspace since $(0,1,0,0), (0,0,1,0) \in U_2$ but $(0,1,0,0)+(0,0,1,0) \in U_2$ (not closed under addition)

c. Yes. Consider $T: \mathbb{R}^2 \to \mathbb{R}^4$ given by $T(s,t) = (s, t+s, t-s, 3t+2s)$. Then, T is linear since

$$\begin{aligned} T(\lambda(s_1,t_1)+(s_2,t_2)) &= T(\lambda s_1+s_2, \lambda t_1+t_2) = (\lambda s_1+s_2, \lambda t_1+t_2+\lambda s_1+s_2, \lambda t_1+t_2-\lambda s_1-s_2, 3(\lambda t_1+t_2)+2(\lambda s_1+s_2)) \\ &= \lambda(s_1, t_1+s_1, t_1-s_1, 3t_1+2s_1) + (s_2, t_2+s_2, t_2-s_2, 3t_2+2s_2) = \lambda T(s_1,t_1)+T(s_2,t_2). \end{aligned}$$

But, $U_3 = \operatorname{Im} T$ and we showed in class that $\operatorname{Im} T$ is a subspace for any linear map.

5\. a) Let $p, q \in \mathbb{R}[x]$ and $\lambda \in \mathbb{R}$. Then, $T(\lambda p+q) = x^2 \cdot (\lambda p+q) = \lambda x^2 \cdot p + x^2 q = \lambda T(p)+T(q)$.

b) We show that $\operatorname{Ker} T = \{0\}$. If $p \in \operatorname{Ker} T$ with $p(x) = a_m x^m + \dots + a_1 x + a_0$, then

$$0 = T(p) = x^2 \cdot (a_m x^m + \dots + a_1 x + a_0) = a_m x^{m+2} + \dots + a_1 x^3 + a_0 x^2 \Rightarrow \forall j,\ a_j = 0 \Rightarrow p = 0.$$

c) We show that $\operatorname{Im} T = \{ p \in \mathbb{R}[x] ;\ p = a_m x^m + \dots + a_3 x^3 + a_2 x^2$ with $a_j \in \mathbb{R}$ and $m \geq 2 \}$. If $p \in \mathbb{R}[x]$ with $p = a_m x^m + \dots + a_1 x + a_0 \Rightarrow T(p) = a_m x^{m+2} + \dots + a_1 x^3 + a_0 x^2$ and conversely, for $p = a_m x^m + \dots + a_3 x^3 + a_2 x^2$ with $m \geq 2$, then if $q = a_m x^{m-2} + \dots + a_3 x + a_2$, one has $T(q) = p$.

6. Let $\lambda_1, \dots, \lambda_k \in \mathbb{K}$ be such that $\lambda_1 T(u_1) + \dots + \lambda_k T(u_k) = 0$. Then,

$$0 = \lambda_1 T(u_1) + \dots + \lambda_k T(u_k) \underset{\uparrow \text{ T is linear}}{=} T(\lambda_1 u_1 + \dots + \lambda_k u_k) \Rightarrow \lambda_1 u_1 + \dots + \lambda_k u_k \in \underset{\uparrow \text{ T is injective}}{\operatorname{Ker} T = \{0\}}$$

$\Rightarrow \lambda_1 u_1 + \dots + \lambda_k u_k = 0 \Rightarrow \lambda_1 = \dots = \lambda_k = 0$ since $u_1, \dots, u_k$ are linearly independent.

7. First, if $v \in \operatorname{span}\{v_1, v_2\}$ then $v = \lambda_1 v_1 + \lambda_2 v_2 = \lambda_1(-v_2 - v_3) + \lambda_2 v_2 = (\lambda_2 - \lambda_1) v_2 + (-\lambda_1) v_3 \in \operatorname{span}\{v_2, v_3\}$.
In the same way, if $v \in \operatorname{span}\{v_2, v_3\}$, then $v = \lambda_1 v_2 + \lambda_2 v_3 = \lambda_1 v_2 + \lambda_2(-v_1 - v_2) = (-\lambda_2) v_1 + (\lambda_1 - \lambda_2) v_2 \in \operatorname{span}\{v_1, v_2\}$.

8 a. We check the 3 necessary conditions for W to be a subspace.

(i) $0 \in W$ since if $(y_1, \dots, y_n) \in U$, then $0 \cdot y_1 + \dots + 0 \cdot y_n = 0$

(ii) If $w_1 = (y_1, \dots, y_n) \in W$ and $w_2 = (z_1, \dots, z_n) \in W$, for $u = (x_1, \dots, x_n) \in U$, one has

$$x_1 \cdot (y_1 + z_1) + \dots + x_n \cdot (y_n + z_n) = (x_1 y_1 + \dots + x_n y_n) + (x_1 z_1 + \dots + x_n z_n) = 0 + 0 = 0$$

$\Rightarrow w_1 + w_2 \in W$.

(iii) If $\lambda \in \mathbb{R}$ and $w = (y_1, \dots, y_n) \in W$, for $u = (x_1, \dots, x_n) \in U$, one has

$$x_1 \cdot (\lambda y_1) + \dots + x_n (\lambda y_n) = \lambda (x_1 y_1 + \dots + x_n y_n) = \lambda \cdot 0 = 0$$

$\Rightarrow \lambda w \in W$

b. We show that $\{\overset{w}{(1,1,1)}\}$ is a basis of W.

First, $u_1 = (1, 0, -1)$, $u_2 = (0, 1, -1)$ is a basis of U (this was done in class).

Now, $w \in W$ since if $u \in U$, then $u = \lambda_1 u_1 + \lambda_2 u_2 = (\lambda_1, \lambda_2, -\lambda_1 - \lambda_2)$ and

$\lambda_1 \cdot 1 + \lambda_2 \cdot 1 + (-\lambda_1 - \lambda_2) \cdot 1 = 0 \Rightarrow w \in W$.

Let $w' = (y_1, y_2, y_3) \in W$. Then,

$$1 \cdot y_1 + 0 \cdot y_2 + (-1) \cdot y_3 = 0 \Rightarrow y_1 = y_3 \quad \text{and} \quad 0 \cdot y_1 + 1 \cdot y_2 + (-1) y_3 = 0 \Rightarrow y_2 = y_3$$

$\Rightarrow y_1 = y_2 = y_3 \Rightarrow w' = y_1 \cdot (1,1,1) = y_1 \cdot w \Rightarrow w' \in \operatorname{span}\{w\}$

Thus, w spans W and $\{w\}$ is a L.I. list $\Rightarrow \{w\}$ is a basis of W.