midterm
math133, linear algebra and geometry
summer 2023

Justify all your claims rigorously. This exam is worth 34% of the grade for math133. Allotted time is 2 hours.

1. (4 points) Consider the following three matrices :

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & -3 \\ 0 & 1 & -2 \\ 2 & 3 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} -2 & -1 & 1 \\ 1 & -3 & 4 \end{pmatrix}$$

Compute $AB + C$.

2. (5 points) Consider the following matrix :

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & 2 \\ \sqrt{2} & 2 & -3 \end{pmatrix}$$

- **a. (3)** Find elementary matrices $E_1, \ldots, E_k$ such that the product $(E_k \ldots E_1) \cdot A$ is in row echelon form.
- **b. (2)** Find $x \in \mathbf{M}_{3\times 1}(\mathbb{R})$ such that

$$Ax = \begin{pmatrix} -2 \\ 4 \\ -2 + 2\sqrt{2} \end{pmatrix}.$$

3. (5 points) Consider the following matrix :

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix} \in \mathbf{M}_{2\times 3}(\mathbb{R})$$

- **a. (2)** Find a matrix $B \in \mathbf{M}_{3\times 2}(\mathbb{R})$ such that $AB = I_2$.
- **b. (1)** Find a matrix $x \in \mathbf{M}_{3\times 1}(\mathbb{R})$ with $x \neq 0$ such that $Ax = 0$.
- **c. (2)** Is there a matrix $C \in \mathbf{M}_{3\times 2}(\mathbb{R})$ such that $CA = I_3$? If so, give such a matrix and prove that it satisfies $CA = I_3$. If not, explain why.

4. (5 points) A square matrix $P \in \mathbf{M}_{n\times n}(\mathbb{K})$ is called an idempotent if $P^2 = P$. For example, both the identity matrix and the zero matrix are idempotents.

- **a. (2)** Give an example of a matrix $(p_{ij}) \in \mathbf{M}_{2\times 2}(\mathbb{R})$ which is an idempotent and which has no zero entry, i.e. $p_{ij} \neq 0$ for all i, j.
- **b. (3)** Show that if P is an idempotent, then $Q := I_n - P$ is an idempotent and that they satisfy $PQ = QP = 0$.

5. (5 points) Let $c \in \mathbb{R}$. Define

$$U = \{(x, y, z) \in \mathbb{R}^3 \ ; \ cx + \pi y + z = c\}$$

a. (3) When is U a subspace of $\mathbb{R}^3$?

b. (2) When is U not a subspace of $\mathbb{R}^3$?

6. (5 points) Let V be a $\mathbb{K}$-vector space and let $U, W \subset V$ be two subspaces. Define

$$U + W = \{u + w \in V \ ; \ u \in U \text{ and } w \in W\}.$$

Show that $U + W$ is a subspace of V.

7. (5 points) For $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$, define

$$\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}.$$

Show that for any invertible matrix $P \in \mathbf{M}_{n \times n}(\mathbb{K})$ and for $m \in \mathbb{Z}_{\geq 0}$, one has

$$(P \operatorname{diag}(\lambda_1, \ldots, \lambda_n) P^{-1})^m = P \operatorname{diag}(\lambda_1^m, \ldots, \lambda_n^m) P^{-1}.$$