Midterm . solutions

1. By direct computation:

$$AB = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & -3 \\ 0 & 1 & -2 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1+1+0+0 & 1-1+2+0 & 2+3-4+0 \\ 2-3+0-2 & 2+3+0-3 & 4-9+0+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ -3 & 2 & -4 \end{pmatrix}$$

and

$$AB + C = \begin{pmatrix} 2 & 2 & 1 \\ -3 & 2 & -4 \end{pmatrix} + \begin{pmatrix} -2 & -1 & 1 \\ 1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ -2 & -1 & 0 \end{pmatrix}.$$

2.a. We use the GJ algo:

$E_1 = T(1,2)$

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_1 A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ \sqrt{2} & 2 & -3 \end{pmatrix}$$

$E_2 = RS(3,1;-\sqrt{2})$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sqrt{2} & 0 & 1 \end{pmatrix} \qquad E_2 E_1 A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 2+\sqrt{2} & -3-2\sqrt{2} \end{pmatrix}$$

$E_3 = RS(3,2;-(2+\sqrt{2}))$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(2+\sqrt{2}) & 1 \end{pmatrix} \qquad E_3 E_2 E_1 A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrices E_1, E_2, E_3 are as required since $E_3E_2E_1A$ is in ref (even in rref here).

1.b. We have, for $M = E_3E_2E_1$,

$$Ax = \begin{pmatrix} -2 \\ 4 \\ -2-2\sqrt{2} \end{pmatrix} \Rightarrow MAx = M \cdot \begin{pmatrix} -2 \\ 4 \\ -2-2\sqrt{2} \end{pmatrix}$$

But,

$$M = E_3E_2E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(2+\sqrt{2}) & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sqrt{2} & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2-\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -\sqrt{2} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2-\sqrt{2} & -\sqrt{2} & 1 \end{pmatrix}$$

$$\Rightarrow M \cdot \begin{pmatrix} -2 \\ 4 \\ -2+2\sqrt{2} \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ \star \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$

$$\star = -2 \cdot (-2-\sqrt{2}) + 4(-\sqrt{2}) + (-2+2\sqrt{2}) = 4 + 2\sqrt{2} - 4\sqrt{2} - 2 + 2\sqrt{2} = 2$$

The system now becomes

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \Rightarrow x_3 = 2,\ x_2 = -2+2x_3 = 2 \text{ and } x_1 = 4 + x_2 - 2x_3 = 4+2-4 = 2$$

3.a. We want $B = (b_{ij})$ with $AB = I_2$. Thus,

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The first system is

$$\begin{cases} b_{11} + b_{31} = 1 \\ 2b_{11} + b_{21} - b_{31} = 0 \end{cases}$$

and we let $b_{31} = 1 \Rightarrow b_{11} = 0$ and $b_{21} = 1$. The second system is

$$\begin{cases} b_{12} + b_{32} = 0 \\ 2b_{12} + b_{22} - b_{32} = 1 \end{cases}$$

and we let $b_{12} = 1$, $b_{32} = -1$ and $b_{22} = -2$. Thus,

$$B = \begin{pmatrix} 0 & 1 \\ 1 & -2 \\ 1 & -1 \end{pmatrix}$$

does the required job.

3b. This is equivalent to solving

$$\begin{cases} x_1 + x_3 = 0 \\ 2x_1 + x_2 - x_3 = 0 \end{cases}$$

and we can choose $x_1 = 1$, $x_3 = -1$ and $x_2 = -3$

3.c. There is no such matrix. Otherwise, given the $x \in M_{3\times 1}$ found in 3b, we would have

$$\begin{aligned}(CA)\cdot x &= I_3 \cdot x = x \\ &= C\cdot (Ax) = C \cdot 0 = 0\end{aligned}$$

and $x \neq 0$.

4a. One could take $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ since $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/4+1/4 & 1/4+1/4 \\ 1/4+1/4 & 1/4+1/4 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

4b. One has $Q^2 = (I_n - P)^2 = (I_n - P)(I_n - P) = I_n \cdot I_n - I_n \cdot P - P \cdot I_n + P^2 = I_n - 2P + P^2 = I_n - P = Q$ since $P^2 = P$. Also, $PQ = P(I_n - P) = P - P^2 = P - P = 0$ and $QP = (I_n - P)P = P - P^2 = P - P = 0$, as required.

5.a. U is a subspace when $c = 0$ since in those case, the set becomes $\{(x,y,z) ;\ \pi \cdot y + z = 0\}$ and

i) $0 \in U$ since $\pi \cdot 0 + 0 = 0$.

ii) If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in U \Rightarrow \pi \cdot (y_1 + y_2) + (z_1 + z_2) = (\pi y_1 + z_1) + (\pi y_2 + z_2) = 0 + 0 = 0$.
$\Rightarrow (x_1, y_1, z_1) + (x_2, y_2, z_2) \in U$.

iii) If $\lambda \in \mathbb{R}$ and $(x,y,z) \in U \Rightarrow \pi \cdot \lambda y + \lambda z = \lambda(\pi y + z) = \lambda \cdot 0 = 0 \Rightarrow \lambda(x,y,z) \in U$.

5b. U is not a subspace when $c \neq 0$ since in those cases, $c \cdot 0 + \pi \cdot 0 + 0 = c$ is never verified $\Rightarrow (0,0,0) \notin U$, breaking a necessary condition for U to be a subspace.

6. We verify the three conditions:

(i) Since $0 \in U, W \Rightarrow 0 = 0 + 0 \in U + W$.

(ii) Let $v_1, v_2 \in U + W$. Then, $\exists u_1, u_2 \in U$ and $w_1, w_2 \in W$ s.t. $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$. Thus,
$v_1 + v_2 = (u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2) \in U + W$.

(iii) Let $\lambda \in \mathbb{K}$ and $v \in U + W \Rightarrow \exists u \in U$ and $\exists w \in W$ s.t. $v = u + w$. Thus, $\lambda \cdot v = \lambda \cdot (u + w) = \lambda u + \lambda w \in U + W$.

7. One has that $\mathrm{diag}(\lambda_i) \cdot \mathrm{diag}(\mu_i) = \mathrm{diag}(\lambda_i \mu_i)$ by a direct computation. Now,

$$(P \cdot \mathrm{diag}(\lambda_i) \cdot P^{-1})^m = P \cdot \mathrm{diag}(\lambda_i) \cdot \underbrace{P^{-1} \cdot P}_{I_n} \cdot \mathrm{dia}(\lambda_i) \cdot \underbrace{P^{-1} \cdot P}_{I_n} \cdot \mathrm{diag}(\lambda_i) P^{-1} \ldots P\, \mathrm{diag}(\lambda_i) P^{-1}$$

$$= P \cdot \underbrace{\mathrm{diag}(\lambda_i) \cdot \mathrm{diag}(\lambda_i) \cdot \ldots \cdot \mathrm{diag}(\lambda_i)}_{m\text{-times}} P^{-1} = P\, \mathrm{diag}(\lambda_i^m) P^{-1}$$

and the result follows