practice final

math133, linear algebra and geometry
summer 2023

Justify all your claims rigorously.

1. Find $x_1, x_2, x_3, x_4 \in \mathbb{R}$ satisfying the following system of equations :

$$\begin{cases} x_1 + 3x_2 + x_3 + x_4 & = 2 \\ x_1 - x_2 + x_3 - x_4 & = 2 \\ 3x_2 + 4x_3 + 3x_4 & = 4 \\ -x_1 + x_2 + 2x_3 + 2x_4 & = 1 \end{cases}$$

2. Let

$$\mathscr{B} = \{(1, 1, -2), (4, 2, -1), (2, 1, -1)\}$$
$$\mathscr{C} = \{(0, 0, 1, 2), (0, 0, 1, 3), (-2, 3, 0, 0), (-1, 1, 0, 0)\}$$

be bases of $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively (you can assume that they indeed are bases). Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3, x_1 - x_2 + 2x_3, x_1 - x_2 + x_3, 2x_1 + 2x_2 + 2x_3).$$

a. Show that T is linear.

b. Is T injective ?

c. What is the dimension of the image of T ?

d. Compute the matrix representation of T in bases $\mathscr{B}$ and $\mathscr{C}$, i.e. $\underset{\mathscr{C} \leftarrow \mathscr{B}}{[T]}$.

3. Let U, V, W be $\mathbb{K}$-vector spaces and let $T : U \to V$ and $S : V \to W$ be linear maps. Show that if T is surjective, then $\operatorname{Im}(S \circ T) = \operatorname{Im} S$.

4. Consider the following subsets of $\mathbb{R}[x]$:

$$U_1 = \{p \in \mathbb{R}[x]\ ;\ p(2) = 0\}$$
$$U_2 = \{p \in \mathbb{R}[x]\ ;\ \text{all the coefficients of } p \text{ are integers}\}$$
$$U_3 = \{p \in \mathbb{R}[x]\ ;\ p = 0 \text{ or } \deg p \geq 3\}$$

Determine which subsets are subspaces and which are not. Justify your answer.

5. Let $V = \mathbb{R}[x]$ be the $\mathbb{R}$-vector space of polynomials with coefficients in $\mathbb{R}$. Consider the map $d : \mathbb{R}[x] \to \mathbb{R}[x]$ given on monomials by

$$d(x^m) = mx^{m-1}$$

and extended linearly. In class, we showed that d is a linear map.

a. What is the kernel of d?

b. What is the image of d?

6. Let U,V be $\mathbb{K}$-vector spaces and let $u_1, \ldots, u_k \in U$ be such that $\operatorname{span}\{u_1, \ldots, u_k\} = U$. Show that if $T : U \to V$ is a surjective linear map, then $\operatorname{span}\{T(u_1), \ldots, T(u_k)\} = V$.

7. Let V be a $\mathbb{K}$-vector space and let v_1, v_2, v_3, v_4 be linearly independent vectors. Show that

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

are linearly independent.

8. Let U be a subspace of $\mathbb{R}^n$.

a. Show that

$$W = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \ ; \ x_1y_1 + x_2y_2 + \cdots + x_ny_n = 0 \text{ for all } (y_1, y_2, \ldots, y_n) \in U\}$$

is a subspace of $\mathbb{R}^n$.

b. Assume that $n = 3$ and that $\dim U = 2$. Show that if $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in U$ are linearly independent and if

$$z_1 = x_2y_3 - x_3y_2, \quad z_2 = x_3y_1 - x_1y_3, \quad z_3 = x_1y_2 - x_2y_1,$$

then the vector $z = (z_1, z_2, z_3)$ is a non-zero vector of the subspace W.