practice midterm

math133, linear algebra and geometry
summer 2023

Justify all your claims rigorously.

1. Consider the following three matrices :

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 5 & -5 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 0 & 3 \\ 3 & 0 & 3 \\ -2 & -2 & -2 \end{pmatrix}$$

Compute $AB + C$.

2. Consider the following matrix :

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 \end{pmatrix}$$

a. Find elementary matrices $E_1, \dots, E_k$ such that the product $(E_k \cdot \dots \cdot E_1) \cdot A$ is in row echelon form.

b. Find $x \in \mathbf{M}_{3\times 1}(\mathbb{R})$ such that

$$Ax = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 4 \end{pmatrix}.$$

3. Find matrices $A \in \mathbf{M}_{3\times 3}(\mathbb{R})$ and $b \in \mathbf{M}_{3\times 1}(\mathbb{R})$ such that the parabola

$$y = a_2 x^2 + a_1 x + a_0$$

passes through the points $(2, 1)$, $(-3, -14)$ and $(-1, -2)$ when the system

$$A \cdot \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = b$$

is satisfied and find the coefficients a_0, a_1, a_2.

4. For $A = (a_{ij}) \in \mathbf{M}_{n\times n}(\mathbb{K})$, define $\operatorname{Tr} A = \sum_{k=1}^{n} a_{kk}$, i.e. $\operatorname{Tr} A$ is equal to the sum of the diagonal entries of A.

a. Show that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for all $A, B \in \mathbf{M}_{n\times n}(\mathbb{K})$.

b. Is it true that $\operatorname{Tr}(AB) = \operatorname{Tr}(A)\operatorname{Tr}(B)$? If so, prove it. If not, give a counterexample.

5. Let $r \in \mathbb{Q}$. Define

$$U = \{(a, b, c, d) \in \mathbb{Q}^4 \; ; \; rab = c + d\}$$

a. When is U a subspace of $\mathbb{Q}^4$?

b. When is U not a subspace of $\mathbb{Q}^4$?

6. Show that the set $V = \mathbb{R}_{>0} = \{x \in \mathbb{R} \; ; \; x > 0\}$, together with the operations "$+$" and "$\cdot$" defined by

$$x + y := xy \qquad \lambda \cdot x := x^\lambda$$

for $x, y \in V$ and $\lambda \in \mathbb{R}$ is an $\mathbb{R}$-vector space.

7. The Fibonnaci sequence is a sequence of integers $f_0, f_1, f_2, \ldots$ defined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. The first few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Observe that this matrix satisfies the following equation :

$$A \cdot \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

a. Consider the matrices

$$P = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 0 \\ 0 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \qquad Q = \begin{pmatrix} \frac{-\sqrt{5}}{5} & \frac{5+\sqrt{5}}{10} \\ \frac{\sqrt{5}}{5} & \frac{5-\sqrt{5}}{10} \end{pmatrix}$$

and compute the products PQ, QP and PDQ.

b. Express f_{2023} in terms of $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$.