Quiz 1. solutions

1a. Using the definition of the product of matrices, we have

$$AB = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -3 \\ -1 & 2 & 5 \\ 2 & -3 & -7 \end{pmatrix} = \begin{pmatrix} 1-2+2 & -1+4-3 & -3+10-7 \\ 3+1-4 & -3-2+6 & -9-5+14 \\ -1-1+2 & 1+2-3 & 3+5-7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$$BA = \begin{pmatrix} 1 & -1 & -3 \\ -1 & 2 & 5 \\ 2 & -3 & -7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1-3+3 & 2+1-3 & 1+2-3 \\ -1+6-5 & -2-2+5 & -1-4+5 \\ 2-9+7 & 4+3-7 & 2+6-7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

1b. If $Ax = b$, it follows that $B\cdot b = B\cdot(A\cdot x) = (B\cdot A)\cdot x = I_3 \cdot x = x$ by associativity of the product and since I_3 is a neutral element for mult. Thus,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 \\ -1 & 2 & 5 \\ 2 & -3 & -7 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+0-3 \\ -4+0+5 \\ 8+0-7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow x_1 = 1,\ x_2 = 1 \text{ and } x_3 = 1.$$

2. We compute directly:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ \frac{-a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{pmatrix} = \begin{pmatrix} \frac{a_{11}a_{22}}{a_{11}a_{22}-a_{12}a_{21}} + \frac{-a_{12}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}a_{12}}{a_{11}a_{22}-a_{12}a_{21}} + \frac{-a_{12}a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \\ \frac{a_{21}a_{22}}{a_{11}a_{22}-a_{12}a_{21}} + \frac{-a_{22}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{-a_{21}a_{12}}{a_{11}a_{22}-a_{12}a_{21}} + \frac{a_{22}a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and similarly,

$$\begin{pmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ \frac{-a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{a_{22}a_{11}}{a_{11}a_{22}-a_{12}a_{21}} + \frac{-a_{12}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{-a_{12}a_{22}}{a_{11}a_{22}-a_{12}a_{21}} + \frac{a_{22}a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ \frac{-a_{21}a_{11}}{a_{11}a_{22}-a_{12}a_{21}} + \frac{a_{11}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{-a_{21}a_{12}}{a_{11}a_{22}-a_{12}a_{21}} + \frac{a_{11}a_{22}}{a_{11}a_{22}-a_{12}a_{21}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is the required result.

3. By a direct computation,

$$K(\theta_1)K(\theta_2) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} = K(\theta_1+\theta_2)$$

as required.