Quiz 2. solutions

1.a. We use the G-J algo :

→ $E_1 = T(1,2)$ swaps rows 1 and 2.

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_1 A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

→ $E_2 = RS(2,1;-1)$ replaces row 2 by row 2 − row 1

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 E_1 A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 0 & 3 \end{pmatrix}$$

→ $E_3 = RS(3,1;-1)$ replaces row 3 by row 3 − row 1

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad E_3 E_2 E_1 A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

1b. Consider

$$M = E_3 E_2 E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

then, one has

$$M\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad M\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \text{ and } M\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

it follows that

$$Ax_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow M \cdot A \cdot x_1 = M \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow x_{13} = 0,\ x_{12} = -1,\ x_{11} = 0$$

$$Ax_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Leftrightarrow M \cdot A x_2 = M \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \\ x_{23} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow x_{23} = -1,\ x_{22} = -(-1+(-1)) = 2,\ x_{21} = 1+2 = 3$$

$$Ax_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Leftrightarrow M A x_3 = M \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_{31} \\ x_{32} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow x_{33} = 1,\ x_{32} = -1,\ x_{31} = -2$$

1c. The matrix B is given by

$$B = \begin{pmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 3 & -2 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

and by a direct computation, $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2. By distributivity of the matrix product on the sum of matrices, it follows that

$$A(x_p + x_h) = Ax_p + Ax_h = b + 0 = b$$

since $Ax_p = b$ and $Ax_h = 0$.