Quiz 3. solutions

1.a. No, they are not linearly independent since $(1)\cdot v_1 + (1) v_2 + (-1) v_3 + (0) v_4 = 0$ is a non-zero linear combination giving zero.

1.b. Yes, they span $\mathbb{R}^3$. In fact, v_1, v_2, v_4 span $\mathbb{R}^3$. This is true since for $(x,y,z) \in \mathbb{R}^3$, one has

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_4 v_4 = (x,y,z)$

$\Leftrightarrow (2\lambda_1, \lambda_1, \lambda_1) + (2\lambda_2, 0, -\lambda_2) + (\lambda_4, \lambda_4, \lambda_4) = (x,y,z)$

$$\Leftrightarrow \begin{cases} 2\lambda_1 + 2\lambda_2 + \lambda_4 = x \\ \lambda_1 + \lambda_4 = y \\ \lambda_1 - \lambda_2 + \lambda_4 = z \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 2 & 2 & 1 & x \\ 1 & 0 & 1 & y \\ 1 & -1 & 1 & z \end{array}\right) \begin{array}{l} L_1 - 2L_2 \mapsto L_2 \\ L_1 - 2L_3 \mapsto L_3 \end{array} \left(\begin{array}{ccc|c} 2 & 2 & 1 & x \\ 0 & 2 & -1 & x-2y \\ 0 & 4 & -1 & x-2z \end{array}\right)$$

$$2L_2 - L_3 \mapsto L_3 \quad \left(\begin{array}{ccc|c} 2 & 2 & 1 & x \\ 0 & 2 & -1 & x-2y \\ 0 & 0 & -1 & x-4y+2z \end{array}\right) \Rightarrow \lambda_4 = -x+4y-2z$$

$$\Rightarrow 2\lambda_2 - \lambda_4 = x - 2y \Leftrightarrow \lambda_2 = \tfrac{1}{2}(x - 2y + \lambda_4) = \tfrac{1}{2}(x - 2y - x + 4y - 2z) = y - z$$

$$\Rightarrow 2\lambda_1 + 2\lambda_2 + \lambda_4 = x \Rightarrow \lambda_1 = \tfrac{1}{2}(x - 2\lambda_2 - \lambda_4) = \tfrac{1}{2}\big(x - 2(y-z) - (-x+4y-2z)\big)$$
$$= \tfrac{1}{2}(2x - 6y + 4z) = x - 3y + 2z.$$

and we find that

$$(x-3y+2z)(2,1,1) + (y-z)(2,0,-1) + (-x+4y-2z)(1,1,1) = (x,y,z).$$

1.c. $\mathcal{B}$ is not a basis since it is not a lin. independent set by 1.a. But, $\{v_1, v_2, v_4\}$ is a basis:

It spans $\mathbb{R}^3$ (see above).

Lin indp: Let $\lambda_1, \lambda_2, \lambda_4 \in \mathbb{R}$ s.t. $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_4 v_4 = 0$. Then, as we saw in 1.b, we have that $\lambda_1, \lambda_2, \lambda_3$ verify the system

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

$\Rightarrow$ They are L.I.

Thus, v_1, v_2, v_4 form a basis by definition.

2. let $u_1, u_2 \in U$ and let $\lambda \in \mathbb{K}$. Then,

$$(S \circ T)(\lambda u_1 + u_2) = S(T(\lambda u_1 + u_2)) = S(\lambda T(u_1) + T(u_2)) = \lambda S(T(u_1)) + S(T(u_2)) = \lambda (S \circ T)(u_1) + (S \circ T)(u_2)$$

which shows that $S \circ T$ is linear by definition.