MATH133 – Summer 2023 – Tutorial 4

Please attempt the following exercises before the next tutorial. I encourage you to try all of them, but no worries of you get stuck!

1. Show that for any square matrix A, if it is invertible, then $AB \neq 0$ for any other non-zero matrix B, where 0 denotes the matrix of all zeroes.

2. Let $v = (v_1, v_2), w = (w_1, w_2)$ be vectors in $\mathbb{R}^2$ and define the following operation:

$$v \cdot w := v_1 w_1 + v_2 w_2 \in \mathbb{R}$$

   (a) Consider the 2 by 2 matrices $P = \begin{pmatrix} -a- \\ -b- \end{pmatrix}$ and $Q = \begin{pmatrix} | & | \\ c & d \\ | & | \end{pmatrix}$, where a, b denote rows, and c, d columns. Show that $PQ = \begin{pmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{pmatrix}$.

   (b) Recall R_θ from tutorial 3. Show that $v \cdot w = R_\theta(v) \cdot R_\theta(w)$.

   (c) Let $|v|$ denote the length of a vector, i.e. $|v| = \sqrt{v_1^2 + v_2^2}$ and let ϕ denote the angle between two vectors. Using elementary geometry, show that $v \cdot w = |v||w| \cos \phi$.

   (d) Show that if v, w are orthogonal $v \cdot w = 0$.

   (e) Let v, w be orthogonal such that $|v| = 1 = |w|$. Find the inverse of the 2 by 2 matrix $\begin{pmatrix} -v- \\ -w- \end{pmatrix}$

3. Show whether or not the following are vector spaces (over $\mathbb{Q}$ or $\mathbb{R}$)

   (a) Polynomials of degree *at most* 20, with coefficients in K, with the usual addition and multiplication.

   (b) Polynomials of degree *exactly* 20, with coefficients in K, with the usual addition and multiplication.

   (c) The set of all increasing functions, i.e. functions f such that if $x \leq y$ then $f(x) \leq f(y)$, with addition defined $(f + g)(x) := f(x) + g(x)$ and multiplication $(\lambda f)(x) := \lambda f(x)$.

   (d) The set of all infinite sequences in $\mathbb{R}$, i.e. $x = (x_0, x_1, x_2, \cdots), x_i \in \mathbb{R}$, with addition defined by

$$(x_0, x_1, x_2, \cdots) + (y_0, y_1, y_2, \cdots) := (x_0 + y_0, x_1 + y_1, x_2 + y_2, \cdots)$$

   and scalar multiplication defined by

$$\lambda(x_0, x_1, x_2, \cdots) := (\lambda x_0, \lambda x_1, \lambda x_2, \cdots)$$

   (e) $V = \mathbb{R}$ with addition defined as $x + y := x \cdot y + y$ and the usual scalar multiplication.