MATH133 – Summer 2023 – Tutorial 6

Please attempt the following exercises before the next tutorial. I encourage you to try all of them, but no worries of you get stuck!

1. Suppose that V is the set of all solutions of the homogeneous system

$$\begin{cases} x_1 + 2x_2 - 2x_3 + 2x_4 - x_5 & = 0 \\ x_1 + 2x_2 - x_3 + 3x_4 - 2x_5 & = 0 \\ 2x_1 + 4x_2 - 7x_3 + x_4 + x_5 & = 0 \end{cases}$$

   Show that V is a subspace of $\mathbb{R}^5$, and find a basis for V.

2. Find a basis for the following vector spaces and compute their dimension:

   (a) $U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, x_3 = 7x_4\} \subseteq \mathbb{R}^5$.

   (b) $M_{m\times n}(\mathbb{R})$.

3. Is $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ in the span of $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 2 \\ 2 \end{pmatrix} \right\}$?

4. Let V be the space of all functions $f : \mathbb{R} \to \mathbb{R}$. Show that $\{e^x, \sin(x), x\}$ are linearly independent.

5. Recall from tutorial 4 the dot product for $v, w \in \mathbb{R}^2$:

$$v \cdot w := v_1 w_1 + v_2 w_2 \in \mathbb{R}$$

   Recall also that the *length* of a vector v is $|v| = \sqrt{v \cdot v}$. Let $\{v, w\}$ be vectors in $\mathbb{R}^2$ such that $|v| = 1 = |w|$ and $v \cdot w = 0$. We call such a set *orthonormal*. Show that v, w are linearly independent.

6. (a) Show that $\mathcal{B} = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$ is a linearly independent set.

   (b) For $v_1, v_2 \in \mathbb{R}$, let $v = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be a vector in the canonical basis of $\mathbb{R}^2$. Express v as a vector in the $\mathcal{B}$-basis, i.e. find $b_1, b_2 \in \mathbb{R}$ such that $v = b_1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + b_2 \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

   (c) We say that multiplying the matrix $T = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ by $v \in \mathbb{R}^2$ *scales* along the first coordinate of a vector v by a factor $\lambda \neq 0$. For example, in the canonical basis, this operation scales along the x-axis. Find a matrix that scales along the diagonal $x = y$ by a factor λ.