MATH133 – Summer 2023 – Tutorial 7

Please attempt the following exercises before the next tutorial. I encourage you to try all of them, but no worries of you get stuck!

1. Let A=$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$.
   (a) Compute $v_1 = A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
   (b) Compute the area of the parallelogram generated by v_1 and v_2, using any formula you want.
   (c) Recall that for a given matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we consider the value $\Delta(A) = a_{11}a_{22} - a_{12}a_{21}$. Compute $\Delta(A)$. What do you notice?

2. Consider the following matrix

$$B = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

   (a) We say that a subspace U is B *invariant* if $BU \subseteq U$ where $BU := \{Bu : u \in U\}$. Find all B invariant subspaces.
   (b) Find invariant subspaces U, V such that $U \cap V = \{0\}$ and $\dim(U) = 1$, $\dim(V) = 2$.
   (c) Find a basis $\mathcal{B}_1$ for U and a basis $\mathcal{B}_2$ for V.
   (d) Recall the dot product for $v, w \in \mathbb{R}^3$:

$$v \cdot w := v_1w_1 + v_2w_2 + v_3w_3 \in \mathbb{R}$$

   Compute $v \cdot w$ for all $v \in \mathcal{B}_1, w \in \mathcal{B}_2$.
   (e) Finally consider

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   and compute BC and CB.

3. *Bonus question:* Let $a, b, u \in \mathbb{R}^3$, and assume that a is not in the line L generated by $b + t \cdot u$. Let x be the point on L such that the distance between a and x is the shortest distance from a to L. Show that u is perpendicular to $x - a$. Conclude the same for the shortest distance from a point to a plane.