Appendix: Brief Euclid

For reference we include abbreviated statements of the most frequently quoted results from Euclid's *Elements*.

Book I. Definitions

1. A *point* is that which has no part.
2. A *line* is length without breadth.
4. A *straight line* lies evenly with its points.
8. A *plane angle* is the inclination of two lines.
10. When the two adjacent angles are equal it is a *right angle*.
15. A *circle* is a line all of whose points are equidistant from one point.
20. A triangle with two equal sides is *isosceles*.
23. *Parallel* straight lines are lines in the same plane that do not meet, no matter how far extended in either direction.

Postulates

1. To draw a line through two points.
2. To extend a given line.
3. To draw a circle with given center through a given point.
4. All right angles are equal.
5. If a line crossing two other lines makes the interior angles on the same side less than two right angles, then these two lines will meet on that side when extended far enough.

Common Notions

1. Things equal to the same thing are equal.
2. Equals added to equals are equal.

3. Equals subtracted from equals are equal.
4. Things which coincide are equal.
5. The whole is greater than the part.

Propositions (preliminary material)

1. To construct an equilateral triangle on a given segment.
2. To draw a segment equal to a given segment at a given point.
3. To cut off a smaller segment from a larger segment.
4. Side–angle–side (SAS) congruence for triangles.
5. The base angles of an isosceles triangle are equal.
6. If the base angles are equal, the triangle is isosceles.
7. It is not possible to put two triangles with equal sides on the same side of a segment.
8. Side–side–side (SSS) congruence for triangles.
9. To bisect an angle.
10. To bisect a segment.
11. To construct a perpendicular to a line at a given point on the line.
12. To drop a perpendicular from a point to a line not containing the point.
13. A line standing on another line makes angles equal to two right angles.
15. Vertical angles are equal.
16. The exterior angle of a triangle is greater than either opposite interior angle.
17. Any two angles of a triangle are less than two right angles.
18. If one side of a triangle is greater than another, then the angle opposite it is greater than the other.
19. If one angle of a triangle is greater than another, then the side opposite it is greater than the other.
20. Any two sides of any triangle are greater than the third.
22. To construct a triangle, given three sides, provided any two are greater than the third.
23. To reproduce a given angle at a given point and side.
24. Two sides equal but included angle greater of two triangles implies base greater.
25. Two sides equal and greater base implies greater angle.
26. Angle–side–angle (ASA) and angle–angle–side (AAS) congruence for triangles.
27. Alternate interior angles equal implies parallel lines.
28. Exterior angle equal to opposite interior, or two interior angles equal to two right angles, implies parallel lines.
29. A line crossing two parallel lines makes alternate interior angles equal.
30. Lines parallel to the same line are parallel.
31. To draw a line parallel to a given line through a given point.
32. Sum of angles of a triangle is two right angles, and exterior angle equals the sum of opposite interior angles.

33. Lines joining endpoints of equal parallel lines are equal and parallel.
34. The opposite sides and angles of a parallelogram are equal.
35. Parallelograms on the same base and in the same parallels are equal.
36. Parallelograms on equal bases in the same parallels are equal.
37. Triangles on the same base in the same parallels are equal.
38. Triangles on equal bases in the same parallels are equal.
39. Equal triangles on the same base on the same side are in the same parallels.
40. Equal triangles on equal bases on the same side are in the same parallels.
41. A parallelogram is twice the triangle on the same base in the same parallels.
42. To construct a parallelogram with a given angle equal to a given triangle.
43. Parallelograms on opposite sides of the diagonal of a parallelogram are equal.
44. To construct a parallelogram with given side and angle equal to a given triangle.
45. To construct a parallelogram with a given angle equal to a given figure.
46. To construct a square on a given segment.
47. (Theorem of Pythagoras) The square on the hypotenuse is equal to the sum of the squares on the sides of a right triangle.
48. If the sum of the squares on two sides equals the square on the third side, the triangle is right.

Book II. Propositions (geometric algebra)

1. The rectangle contained by two lines is the sum of the rectangles contained by one and the segments of the other.
4. The square on the whole line is equal to the squares on its two segments plus twice the rectangle on the two segments.
5. The square on half a line is equal to the rectangle on the unequal segments plus the square of the difference.
6. The rectangle on a line plus an added piece with the added piece, plus the square of half the segment, is equal to the square of the half plus the added piece.
11. To cut a line so that the rectangle on the whole and one segment is equal to the square on the other segment (extreme and mean ratio).
14. To construct a square equal to a given figure.

Book III. Propositions (properties of circles)

1. To find the center of a circle.
2. The segment joining two points of a circle lies inside the circle.
5. If two circles intersect, they do not have the same center.
6. If two circles are tangent, they do not have the same center.
10. Two circles can intersect in at most two points.
11, 12. If two circles are tangent, their centers lie in a line with the point of tangency.
16. The line perpendicular to a diameter at its end is tangent to the circle, and

the angle between the tangent line and the circle is less than any rectilineal angle.

17. To draw a tangent to a circle from a point outside the circle.
18. A tangent line to a circle is perpendicular to the radius at the point of tangency.
19. The perpendicular to a tangent line at the point of tangency will pass through the center of the circle.
20. The angle at the center is twice the angle at a point of the circumference subtending a given arc of a circle.
21. Two angles from points of a circle subtending the same arc are equal.
22. The opposite angles of a quadrilateral in a circle are equal to two right angles.
31. The angle in a semicircle is a right angle.
32. The angle between a tangent line and a chord of a circle is equal to the angle on the arc cut off.
35. If two chords cut each other, the rectangle on the segments of one chord is equal to the rectangle on the segments of the other chord.
36. From a point outside a circle, let a tangent and a secant line be drawn. Then the square of the tangent line is equal to the rectangle formed by the two segments from the point to the circle on the secant line.
37. From a point outside a circle, if two lines cut the circle, so that the square of one is equal to the rectangle formed by the segments of the other, then the first is a tangent line.

Book IV. Propositions (construction of regular polygons)

1. To inscribe a given segment in a circle.
2. To inscribe a triangle, equiangular to a given triangle, in a circle.
3. To circumscribe a triangle, equiangular to a given triangle, around a circle.
4. To inscribe a circle in a triangle.
5. To circumscribe a circle around a triangle.
10. To construct an isosceles triangle whose base angles are twice the vertex angle.
11. To inscribe a regular pentagon in a circle.
12. To circumscribe a regular pentagon around a circle.
15. To inscribe a regular hexagon in a circle.
16. To inscribe a regular 15-sided polygon in a circle.

Book V. Definitions

4. Magnitudes are said to *have a ratio* if either one, being multiplied, can exceed the other.
5. Four magnitudes a, b; c, d are *in the same ratio* if for any whole numbers m, n, we have $ma > nb$ or $ma = nb$ or $ma < nb$ if and only if $mc > nd$ or $mc = nd$ or $mc < nd$ respectively.

Book VI. Propositions (theory of proportion)

1. Triangles of the same height are in the same ratio as their bases.
2. A line is parallel to the base of a triangle if and only if it cuts the sides proportionately.
3. A line from a vertex of a triangle to the opposite side bisects the angle if and only if it cuts the opposite side in proportion to the remaining sides of the triangle.
4. The sides of equiangular triangles are proportional.
5. If the sides of two triangles are proportional, their angles are equal.
6. If two triangles have one angle equal and the sides containing the angle proportional, the triangles will be similar.
8. The altitude from the right angle of a right triangle divides the triangle into two triangles similar to each other and to the whole.
12. To find a fourth proportional to three given lines.
13. To find a mean proportional between two given lines.
16. Four lines are proportional if and only if the rectangle on the extremes is equal to the rectangle on the means.
30. To cut a line in extreme and mean ratio.
31. Any figure on the hypotenuse of a right triangle is equal to the sum of similar figures on the sides of the triangle.

Book X. Propositions (study of irrationals)

1. Given two unequal quantities, if one subtracts from the greater a quantity greater than its half, and repeats this process enough times, there will remain a quantity lesser than the smaller of the two original quantities.
117. (not in Heath, but in Commandino). The diagonal of a square is incommensurable with its side.

Book XI. Definitions

25. A *cube* is a polyhedron made of six equal squares.
26. An *octahedron* is a polyhedron made of eight equal equilateral triangles.
27. An *icosahedron* is a polyhedron made by twenty equal equilateral triangles.
28. A *dodecahedron* is a polyhedron made by twelve equal regular pentagons.

Propositions (solid geometry)

21. The plane angles in a solid angle make less than four right angles.
28. A parallelepiped is bisected by its diagonal plane.
29, 30. Parallelepipeds on the same base and of the same height are equal.
31. Parallelepipeds on equal bases, of the same height, are equal.

Book XII. Propositions (solid geometry)

2. Circles are in the same ratio as the squares of their diameters.
3. A pyramid is divided into two pyramids and two prisms.

5. Pyramids of the same height on triangular bases are in the same ratio as their bases.
7. A prism with a triangular base is divided into three equal triangular pyramids.

Book XIII. Propositions (solid geometry)

7. If at least three angles of an equilateral pentagon are equal, the pentagon will be regular.
10. In a circle, the square on the side of the inscribed pentagon is equal to the square on the side of the inscribed hexagon plus the square on the side of the inscribed decagon.
13. To inscribe a tetrahedron in a sphere.
14. To inscribe an octahedron in a sphere.
15. To inscribe a cube in a sphere.
16. To inscribe an icosahedron in a sphere.
17. To inscribe a dodecahedron in a sphere.
18. (Postscript). Besides these five figures there is no other contained by equal regular polygons.