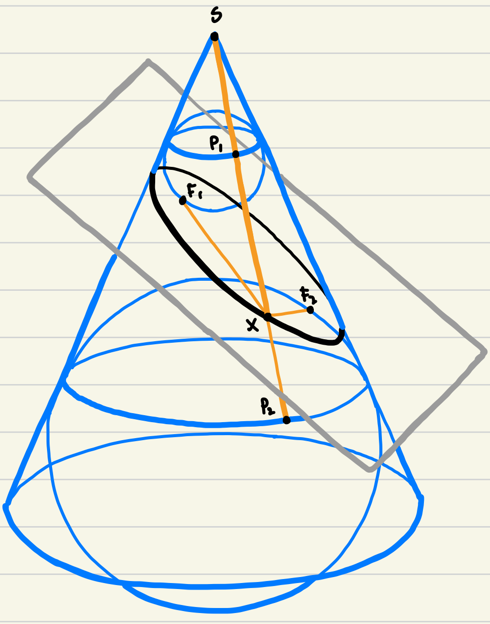
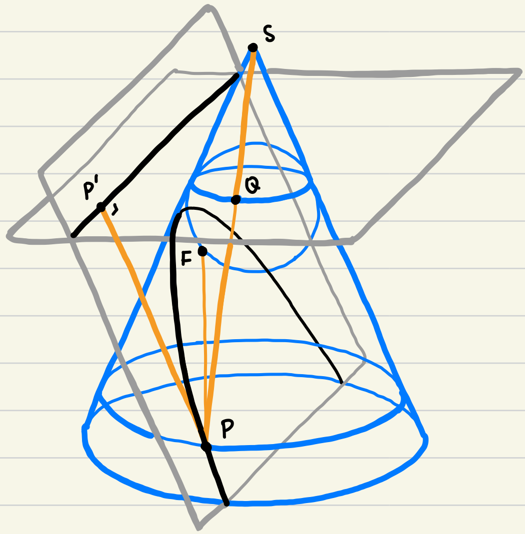
CONIC SECTIONS

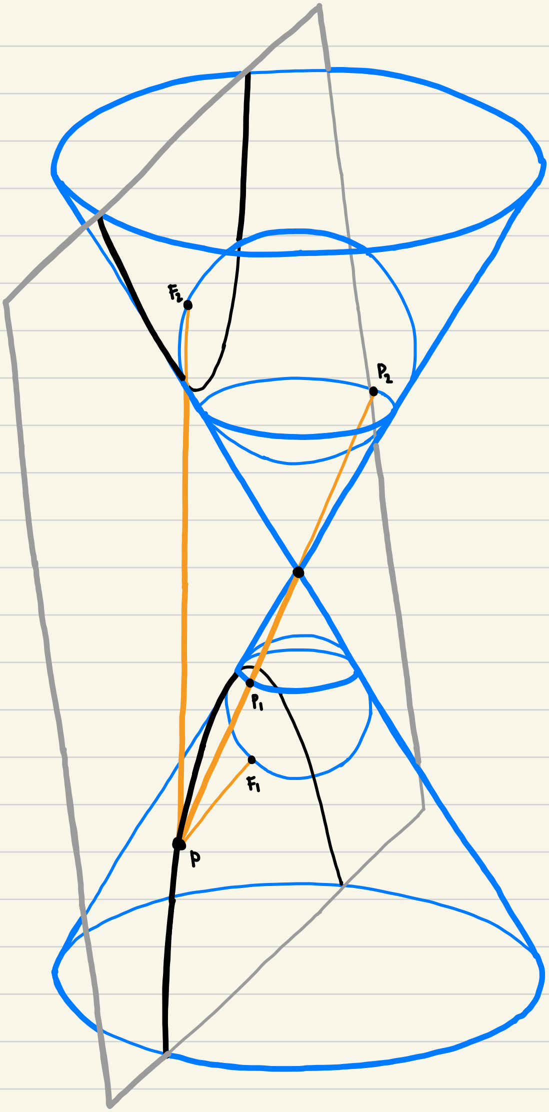
AND THE DANDELIN SPHERES



ELLIPSE



PARABOLA



HYPERBOLA

! Refer to the previous drawing for notation

Properties of the ellipse.

Let P be a point on the ellipse. Then, $|PF_1| = |PP_1|$ and $|PF_2| = |PP_2|$. Thus, $|PF_1| + |PF_2| = |P_1P_2|$ and $|P_1P_2|$ does not depend on P. It is the distance between the two circles of tangency along the cone.

Properties of the parabola.

Consider a point P on the parabola and draw segments PF, PQ where F is the tangent point of the sphere with plane and Q lies on the circle of tangency of the cone with the sphere and on the segment PS. We argued that $|PF| = |PQ|$. Consider a circle passing through P which is parallel to the circle of tangency. Let Q' be the projection of Q on that plane and let P'' be the projection of P' on that plane. Then, we have:

$\angle P'P''P = \angle QQ'P = \frac{\pi}{2}$ $\quad \angle PQQ' = \text{angle of the cone}$ $\quad \angle P''P'P = \text{angle of the plane}$ $\quad |QQ'| = |P'P''| = \text{distance between the two planes}$

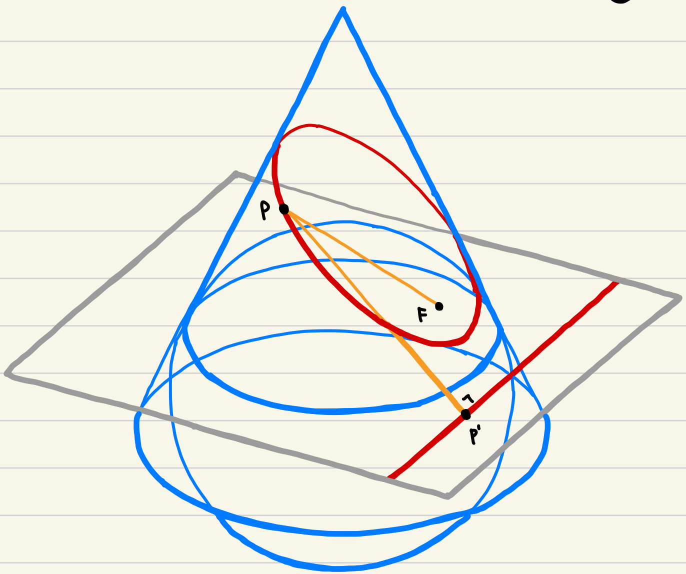
By ASA, PQQ' is congruent to $PP'P''$ $\Rightarrow |PQ| = |PP'|$.

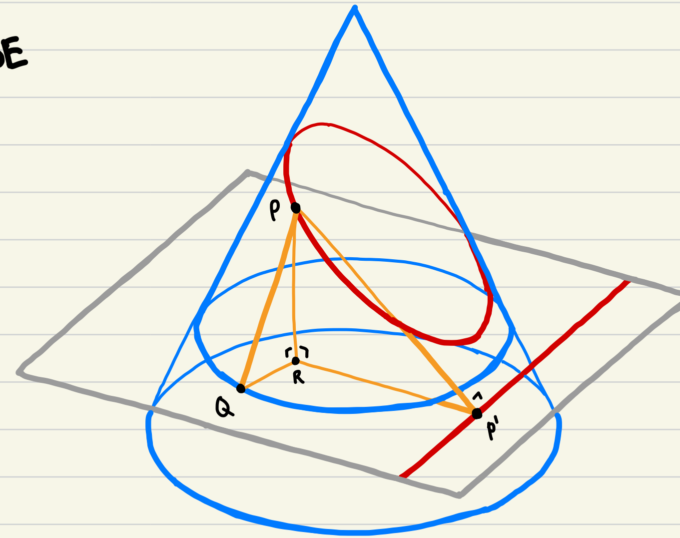
Properties of the hyperbola.

Let P be a point on the hyperbola. The, $|PF_1| = |PP_1|$ and $|PF_2| = |PP_2|$. Thus, $|PF_2| - |PF_1| = |PP_2| - |PP_1| = |P_1P_2|$ and $|P_1P_2|$ does not depend on P. It is the distance between the two circles of tangency along the cone.

ECCENTRICITY

OF THE ELLIPSE





with respect to the axis of revolution of the cone

$\angle QPR = \text{angle of the cone}$ $\quad$ $\angle P'PR = \text{angle of the plane}$

$$\frac{|PR|}{|PQ|} = \cos(\text{angle of the cone}) \qquad \frac{|PR|}{|PP'|} = \cos(\text{angle of the plane}) \qquad \frac{|PF|}{|PP'|} = \frac{|PQ|}{|PP'|} = \frac{\cos(\text{angle of the plane})}{\cos(\text{angle of the cone})}$$