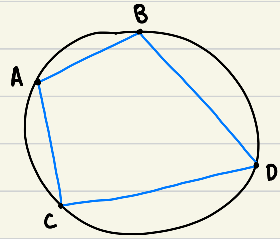
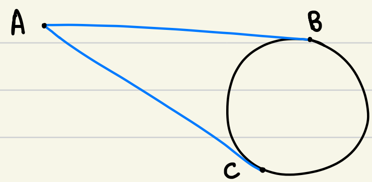
EX 1. A cyclic quadrilateral is a figure with four sides having all its vertices lying on a given circle. Show that if ABCD is a cyclic quadrilateral (having points A, B lying on the same side of the line CD), then opposite angles sum to 2 right-angles. (See [Hart13] 1.5.8 for the converse).

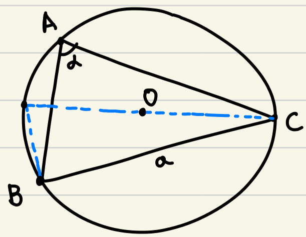


EX 2. Consider a circle and a point A lying outside of the circle. Write B, C for the two distinct points such that AB and AC are tangent to the circle. Show that $|AB| = |AC|$.



EX 3. Show that if ABC is a triangle having circumradius R, sides of length $|BC| = a$, $|AC| = b$, $|AB| = c$ and angles $\angle BAC = \alpha$, $\angle ABC = \beta$, $\angle ACB = \gamma$, then

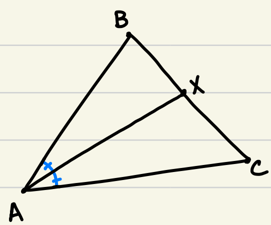
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.$$



This result is called the "extended law of sines".

EX 4. Show that an angle bisector of a triangle cuts the side opposite to that angle in a ratio proportional to the adjacent sides. More precisely:

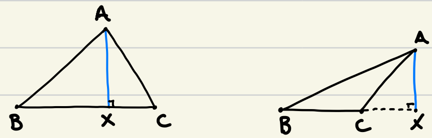
$$\frac{|BX|}{|CX|} = \frac{|AB|}{|AC|}$$



EX 5. Use Ceva's thm and EX 4 to show that the three angle bisectors of a triangle are concurrent.

EX 6. Show that the altitudes of a triangle divide (extended) sides with the following ratio:

$$\frac{|BX|}{|CX|} = \left| \frac{\cos(\beta)\,|AB|}{\cos(\gamma)\,|AC|} \right|$$

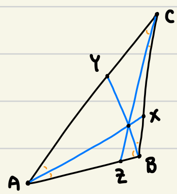


* Make sure you distinguish between the case where ABC is obtuse vs acute. This should account for the absolute value.

EX 7. Use Ceva's thm and EX 6 to show that the three heights of a triangle are concurrent.

EX 8. Prove "Ceva's sine thm": Three cevians AX, BY, CZ are concurrent iff

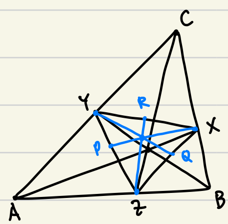
$$\frac{\sin(\angle BAX)}{\sin(\angle CAX)} \cdot \frac{\sin(\angle CBY)}{\sin(\angle ABY)} \cdot \frac{\sin(\angle ACZ)}{\sin(\angle BCZ)} = 1 .$$



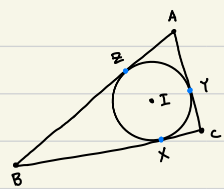
EX 9. Suppose AX, BY, CZ are three concurrent cevians. Let XP, YQ, ZR be three concurrent cevians of triangle XYZ. Show that AP, BQ, CR are concurrent.

Hint: Use EX 8 and the fact that

$$\frac{\sin(\angle BAP)}{\sin(\angle CAP)} \cdot \frac{|AX|}{|AY|} = \frac{|PZ|}{|PY|} .$$



EX 10. Let ABC be a triangle having incenter I. We showed that I is the center of an inscribed circle in ABC, i.e. this circle is tangent to AB, BC, AC. Let X, Y, Z denote the points of tangency of that inscribed



circle on BC, AC and AB respectively (see figure). Show that AX, BY and CZ are concurrent. This point is called the Gergonne point of ABC.

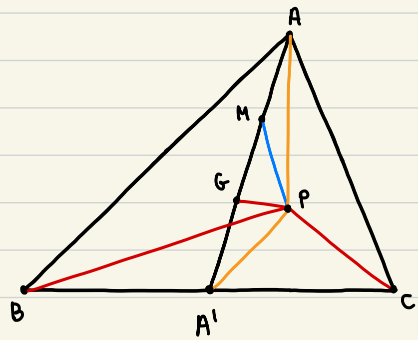
EX 11.

THE EULER LINE
XTREME VERSION

Assume that for any triangle ABC such that A' bisects BC, the following formula holds:

$$2|AA'|^2 = |AB|^2 + |AC|^2 - \frac{1}{2}|BC|^2. \quad \star$$

Let ABC be a triangle, G its centroid and M the midpoint of AG. Let P be any point.



a. By applying $\star$ to the triangles BCP, A'MP and AGP, show that

$$|AP|^2 + |BP|^2 + |CP|^2 - 3|GP|^2 = \frac{1}{2}|BC|^2 + \frac{3}{2}|AG|^2$$

b. Deduce from a. that

$$3(|AP|^2 + |BP|^2 + |CP|^2 - 3|GP|^2)$$
$$= \frac{1}{2}\Big((|AB|^2 + |AC|^2 + |BC|^2) + 3(|AG|^2 + |BG|^2 + |CG|^2)\Big).$$

c. Using $\star$ and b., show that

$$|AB|^2 + |AC|^2 + |BC|^2 = 3(|AP|^2 + |BP|^2 + |CP|^2 - 3|GP|^2).$$

d. Conclude that, if O is the circumcenter of ABC,

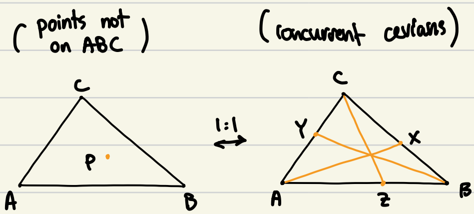
$$|OG|^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

and deduce the distances between the centers O, G, N and H. Add results relating I to O for completeness.

EX 12. Isotomic and isogonal conjugation.

(Before doing this exercise, go take a look at Q4 from the practice midterm.)

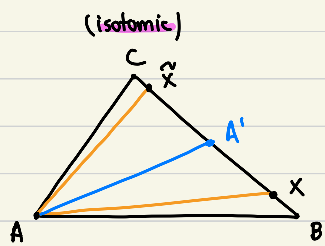
Let ABC be a triangle. We define two transformations of the points not on ABC, namely isotomic conjugation and isogonal conjugation. Recall that given a point P not on ABC, it defines a set of concurrent cevians AX, BY, CZ and vice versa.



Isotomic. Given P not on ABC, let AX, BY, CZ be the concurrent cevians determined by P. Let $\tilde{X}$ be the point on BC such that $\tilde{X} \neq X$ and $|A'X| = |A'\tilde{X}|$ where A' is the midpoint of BC. Define $\tilde{Y}, \tilde{Z}$ analogously.

(i) Show that $A\tilde{X}$, $B\tilde{Y}$ and $C\tilde{Z}$ are concurrent. Their point of concurrence is called the isotomic conjugate of P (denoted $\tilde{P}$).

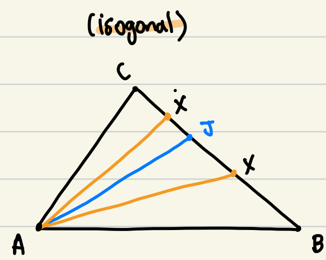
(ii) Show that $(\tilde{\tilde{P}}) = P$.



Isogonal. Given P not on ABC, let AX, BY, CZ be the concurrent cevians determined by P. Let $\dot{X}$ be the point on BC such that $\dot{X} \neq X$ and $\angle JAX = \angle JA\dot{X}$ where AJ is the angle bisector of $\angle BAC$. Define $\dot{Y}, \dot{Z}$ analogously.

(i) Show that $A\dot{X}$, $B\dot{Y}$ and $C\dot{Z}$ are concurrent. Their point of concurrence is called the isogonal conjugate of P (denoted $\dot{P}$).

(ii) Show that $(\dot{\dot{P}}) = P$.



Characterize the points such that $\tilde{P} = P$ and $\dot{P} = P$.