final
math228, classical geometry
fall 2021

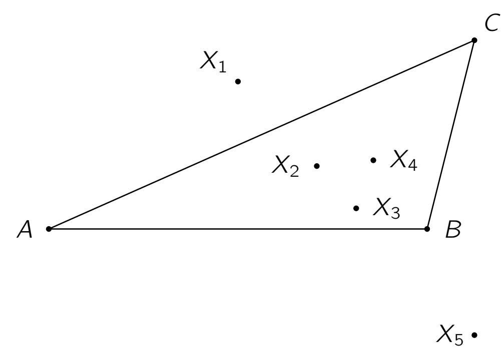
Justify all your claims rigorously. The total score is 70 points.

(5 points) **1.** State Euclid's 5 postulates.

(5 points) **2.** Consider the triangle ABC and the points $X_1, \ldots, X_5$ as in the figure on the right. Indicate which point X_i corresponds to which of the following 5 triangle centers :

- **i.** centroid,
- **ii.** circumcenter,
- **iii.** orthocenter,
- **iv.** incenter,
- **v.** nine-point center.

Justify your answer by using arguments relating the centers together.

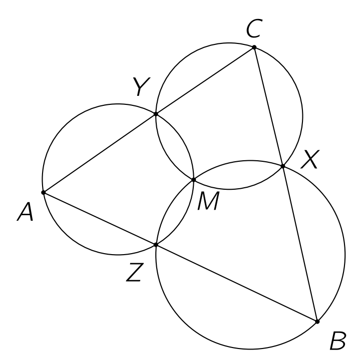


(10 points) **3.** Let ABC be a triangle and let X, Y, Z, be points on BC, AC and AB respectively. Consider three circles $\Gamma_1, \Gamma_2, \Gamma_3$ such that

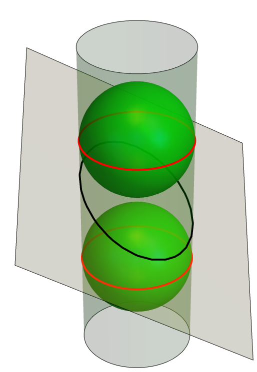
$$A, Y, Z \in \Gamma_1, \quad B, X, Z \in \Gamma_2, \quad C, X, Y \in \Gamma_3,$$

as in the figure on the right. Show that $\Gamma_1, \Gamma_2, \Gamma_3$ have a common point of intersection.

Hint : Let M be the point of intersection of Γ_2 and Γ_3. Show that $AYMZ$ is a cyclic quadrilateral.

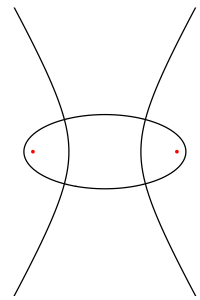


(10 points) **4.** Using an argument analogous to the one of the Dandelin spheres, show that the intersection of a plane and a cylinder (when the intersection is not empty) is an ellipse.
You can assume the existence of spheres tangent to both the cylinder and the plane, as in the figure on the right.

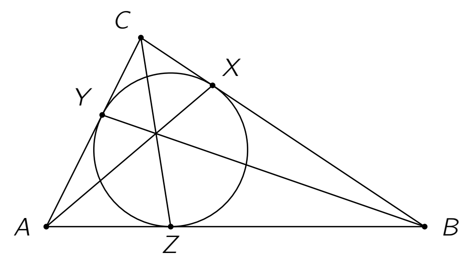


(10 points) **5.** Show that if ϵ is an ellipse and υ is a hyperbola both having foci F_1, F_2, then ϵ and υ intersect at right angles.

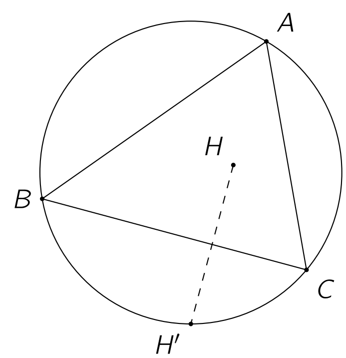
Recall : Given two smooth curves which intersect at a point P, the angle of intersection of the curves at P is given by the angle of intersection of their tangents at P.



(10 points) **6.** Let ABC be a triangle. Consider X, Y, Z the points of tangency of the incircle of ABC with the sides BC, AC and AB respectively. Using Ceva's theorem, show that AX, BY, CZ are concurrent.



(10 points) **7.** Let ABC be a triangle and let H be its orthocenter. Show that the reflection of H with respect to BC lies on the circumcircle of ABC.



(10 points) **8.** Consider an incidence geometry with a relation of betweeness "$\star$". Let A, B, C, D, E be 5 distinct points such that $A \star B \star C$ and $A \star D \star E$ where A, B, D are not collinear. Show that the segments CD and BE intersect.

On the right are the axioms of incidence and the axioms of betweeness.

- *I1. For any two distinct points A, B, there exists a unique line ℓ containing A and B.*
- *I2. Every line contains at least two points.*
- *I3. There exists three noncollinear points.*

- *B1. If $A \star B \star C$, then A, B, C are three distinct points on a line and $C \star B \star A$.*
- *B2. For any two distinct points A, B, there exists a point C such that $A \star B \star C$.*
- *B3. Given three distinct points on a line, one and only one of them is between the other two.*
- *B4. Let A, B, C be three noncollinear points and let ℓ be a line not containing any of A, B, C. If ℓ contains a point D lying between A and B, then it must also contain a point lying between A and C or a point lying between B and C but not both.*