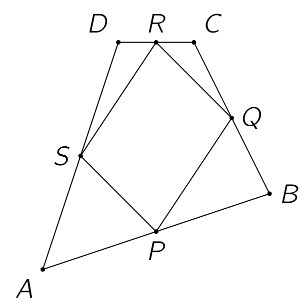
practice midterm
math228, classical geometry
fall 2021

Justify all your claims rigourously.

1. Given two lines intersecting at a point P and a point Q lying on neither of the lines, give a straight-edge compass construction to find a circle tangent to both lines and passing through Q.

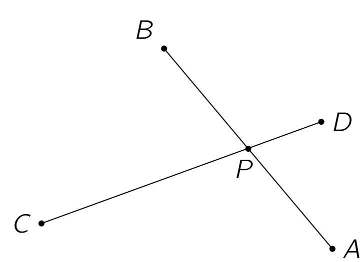
2.



- **a.** Let ABC be a triangle and let A', B', C' be the midpoints of the sides BC, AC, AB respectively. Show that the segment $A'B'$ is parallel to AB.
- **b.** Let $ABCD$ be a quadrilateral and let P, Q, R, S be the midpoints of AB, BC, CD, AD respectively. Using **a**, show that $PQRS$ is a paralellogram.

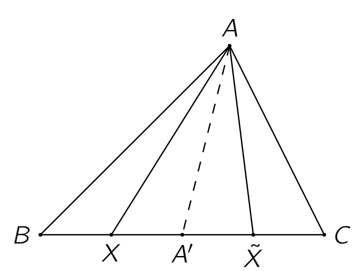
3. Let A, B, C, D be four points such that the segments AB and CD intersect at a point P and such that

$$|AP| \cdot |BP| = |CP| \cdot |DP|.$$



- **a.** Show that the triangles ADP and BCP are similar.
- **b.** Using the characterization of cyclic quadrilaterals, conclude that A, B, C, D all lie on the same circle.

4. Given a triangle ABC and three cevians AX, BY, CZ, let $\tilde{X}$ be the point on BC such that $X \neq \tilde{X}$ and $|A'X| = |A'\tilde{X}|$, where A' is the bisector of BC. Define $\tilde{Y}$ and $\tilde{Z}$ similarly. Apply Ceva's theorem to show that AX, BY and CZ are concurrent if and only if $A\tilde{X}$, $B\tilde{Y}$ and $C\tilde{Z}$ are concurrent.



5. Let ABC be an acute triangle and let DEF be its orthic triangle, where D, E, F are the feet of the altitudes at A, B and C respectively. Denote the orthocenter of ABC by H.

a. Using the characterization of cyclic quadrilaterals, show that the points A, E, F and H all lie on a common circle.

b. First, show using **a** that $\angle EFH = \angle EAH$. Also show that $\angle CAD = 90° - \angle ACD$ and conclude that $\angle EFH = 90° - \angle ACD$.

c. From **b**, conclude that CF is the angle bisector of $\angle DFE$.

d. Show that the orthocenter of an acute triangle coincides with the incenter of its orthic triangle.

e. Was the hypothesis that ABC is acute necessary ? If so, where was it used in the preceeding argument ?